CONVERGENCE AND STABILITY OF BOUNDED MODULUS DISTORTION MAPPINGS

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It is known (see [1, 2]) that a quasiconformal mapping of the space $\mathbb{R}^n$ is characterized by a relation between the moduli of families of curves $\Gamma$ and the moduli of their images $\Gamma^*$; the existence of a constant $K > 0$, for which for all families $\Gamma$ we have the double inequality

$$K^{-1}M(\Gamma) \leq M(\Gamma^*) \leq KM(\Gamma),$$

(1)
defines the $K$-quasiconformality of the mapping. If, in particular, we consider the restriction of a $K$-quasiconformal mapping of the space $\mathbb{R}^n$ to some continuum $\Sigma$, then for any pair of subcontinua $E, F \subset \Sigma$, the double inequality (1) is satisfied for the family $\Gamma$ of all curves in $\mathbb{R}^n$, joining $E$ and $F$. The fact that the "image" of this family $\Gamma$ under the mapping $f$ (this is the family $\Gamma^*$ of all curves in $\mathbb{R}^n$ joining $f(E)$ and $f(F)$) is also well defined in the case when the mapping $f$ is only defined on the continuum $\Sigma$ allows us to consider the class of homeomorphic mappings from $\Sigma$ to $\mathbb{R}^n$, for which inequality (1) is satisfied for any choice of subcontinua $E, F \subset \Sigma$. This class, in particular, contains the restrictions to $\Sigma$ of all the $K$-quasiconformal mappings of the space. The study of the mappings in this class, which we call BMD-homeomorphisms (i.e., bounded modulus distortion homeomorphisms), is mainly connected with the problem of extending a homeomorphism from the continuum $\Sigma$ to a quasiconformal mapping of the whole space. The original idea of studying these mappings came from P. P. Belinskii. There exist examples where BMD-homeomorphisms of the continuum $\Sigma$ cannot be continued to the whole space, or even to a neighborhood of $\Sigma$. In this connection the study of BMD-homeomorphisms is of interest in its own right. Some of their properties were established in [3-5].

The main result of this article is a convergence theorem for sequences of BMD-homeomorphisms and a compactness theorem for normed families of BMD-homeomorphisms. These results are analogous to the well-known convergence theorems for spatial quasiconformal mappings (see [6]). When studying the convergence of mappings with different domains of definition, we shall usually have in mind the idea of convergence to a kernel in the sense of Carathéodory, which allows us to consider the uniform convergence of mappings on compact subsets of the kernel (see [7, 8]). Since BMD-homeomorphisms are defined on different continua, which in general are not regions in $\mathbb{R}^n$, then in studying problems of convergence it seems to be more convenient to use the Hausdorff distance between continua, and to regard the convergence of mappings as the convergence of their graphs in the metric space of continua. Since convergence of continua in the Hausdorff metric is connected with the convergence of their complements to a kernel, then the results of Secs. 2 and 3 can also be stated in terms of the convergence of open sets to a kernel.

The stability of quasiconformal mappings, i.e., their closeness to conformal mappings corresponding to the closeness of the coefficient of quasiconformality to one, has been studied in various forms of the problem, in connection with different concepts of the closeness of mappings, by Lavrent'ev, Belinskii, Reshetnyak, and others (see, for example, [9-13]). The same problem can also be formulated for BMD-homeomorphisms, considering them on a fixed continuum or, in the more general situation, on a family of continua. In Sec. 4 we establish a qualitative theorem on stability (i.e., without indicating concrete estimates for closeness) for BMD-homeomorphisms, defined on a fixed locally connected continuum $\Sigma$ with an estimate for the closeness of mappings in the metric of the space $C(\Sigma)$. We note that we do not always have stability for quasiconformal mappings on a closed spatial region. Examples of such regions have been indicated by Reshetnyak. In the case of BMD-homeomorphisms the situation, as we shall see, is slightly simpler. In the same section we shall establish that for BMD-homeomorphisms, considered on a uniformly locally connected family of continua, stability is uniform, both in relation to the family of these continua, and in relation to the norming of mappings. In the case of BMD-homeomorphisms defined on a k-dimensional sphere.
in $\mathbb{R}^n$, a stability theorem is established in a form independent of the norming, corresponding to the manner in which Reshetnyak formulated the stability problem in [14].

1. Preliminary Information

The symbol $f: A \to B$ will denote a mapping from the set $A$ to the set $B$. The image of the set $U$ under the mapping $f: A \to B$ is the set $\{f(x) : x \in U \cap A\}$. In particular, if $U \cap A = \emptyset$, then $f(U) = \emptyset$. For $A' \subset A$, the symbol $f|A'$ denotes the restriction of the mapping $f$ to the set $A'$; id is the identity mapping; the symbols $f \circ g$ or $fg$ denote the superposition of mappings; and the symbol $f^{-1}$ denotes the inverse mapping of $f$. For $B' \subset B$, the symbol $f^{-1}(B')$ denotes the complete inverse image of the set $B'$ under the mapping $f$.

A continuum in the metric space $\mathcal{X}$ is any nonempty connected compact subset of it. A continuum containing more than one point is called nondegenerate. The distance between the points $x$ and $y$ in the metric space $\mathcal{X}$ is written $d(x; y)$, and the distance from the set $A \subset \mathcal{X}$ to the point $x \in \mathcal{X}$ is the quantity $\rho(x; A) = \inf_{y \in A} d(x; y)$. The distance between the sets $A, B \subset \mathcal{X}$ is defined by the formula

$$|A; B|_{\mathcal{X}} = \inf_{x \in A; y \in B} |x; y|_{\mathcal{X}}.$$ 

The diameter of the set $A \subset \mathcal{X}$ is the quantity

$$\text{diam}_{\mathcal{X}} A = \sup_{x, y \in A} |x; y|_{\mathcal{X}}.$$ 

Moreover, we assume that $\text{diam}_{\mathcal{X}} \emptyset = 0$.

We now describe the metric spaces used in this article.

The space $\mathbb{R}^n$ is $n$-dimensional Euclidean space, and the distance between points in $\mathbb{R}^n$ is denoted by $||x - y||$.

The space $\mathbb{R}^n$ is $n$-dimensional Möbius space with the spherical metric. This can be isometrically identified with an $n$-dimensional sphere in $\mathbb{R}^{n+1}$, with a metric induced by the Euclidean metric. In the case of $\mathbb{R}^n$ we shall omit the index in the notation of the distances between points, a point and a set, between sets, and in the notation of the diameter of a set.

The space $T = \mathbb{R}^n \times \mathbb{R}^n$ is the direct product of metric spaces. The mappings $\text{pr}_1: (x; y) \to x$ and $\text{pr}_2: (x; y) \to y$ are canonical projections. The distance between the points $a = (a_1; a_2)$ and $b = (b_1; b_2)$ in $T$ is defined by the formula

$$|a; b|_T = |a_1; b_1|^2 + |a_2; b_2|^2.$$ 

The space $T$ is compact.

The space $C(\Sigma)$, where $\Sigma$ is a nondegenerate continuum in $\mathbb{R}^n$, is the space of all continuous mappings $f: \Sigma \to \mathbb{R}^n$ with the distance

$$|f; g|_{C(\Sigma)} = \max_{x \in \Sigma} |f(x); g(x)|.$$ 

Convergence in $C(\Sigma)$ is uniform convergence on $\Sigma$.

The space $\text{comp} \mathcal{X}$ is the space of all compact subsets of the compact metric space $\mathcal{X}$, with the following metric (the Hausdorff distance):

$$|A; B|_{\text{comp} \mathcal{X}} = \max \{\sup_{x \in A} |x; B|_{\mathcal{X}}; \sup_{y \in B} |y; A|_{\mathcal{X}}\}$$

(see [15], Sec. 21). The space $\text{comp} \mathcal{X}$ is uniform convergence on $\Sigma$.

The space $\text{cont} \mathcal{X}$ is the space of all continua in the compact metric space $\mathcal{X}$, considered as a subspace of $\text{comp} \mathcal{X}$. The space $\text{cont} \mathcal{X}$ is compact (see [16], Sec. 46, Theorem 14). The canonical projections $\text{pr}_1: T \to \mathbb{R}^n$ generate a continuous mapping $\text{pr}_1: \text{cont} T \to \text{cont} \mathbb{R}^n$ of metric spaces; in particular, if the sequence $\tilde{E}_k = \Gamma$ converges in $\text{cont} T$, then the sequences $\text{pr}_1 \tilde{E}_k \to \text{pr}_1 \Gamma$ and $\text{pr}_2 \tilde{E}_k \to \text{pr}_2 \Gamma$ converge in $\text{cont} \mathbb{R}^n$.

For the pair of continua $E, F \subset \mathbb{R}^n$ we denote by the symbol $M(E; F)$ the modulus of the family of all the curves in $\mathbb{R}^n$ which join $E$ and $F$ (for the definition and properties, see...