SMOOTHNESS OF ROOTS OF POLYNOMIALS DEPENDING ON PARAMETERS

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In studying a number of questions (cf., e.g., [1, 2]) it becomes necessary to investigate the smoothness in \( \mathbb{R}^n \) of the roots of the polynomial \( X^n + a_1(\xi)X^{n-1} + \ldots + a_m(\xi) \) \((a_\xi(\xi) \in \mathbb{C})\). It is well known that the roots have the same smoothness as the coefficients if the multiplicity of the roots is constant. At points where multiplicity changes the roots are, in general, not smooth. For example, the roots of the polynomial \( X^2 + t \) are not differentiable at \( t = 0 \). In the present note it is proved that if all the roots are real, then smoothness is not lost completely.

We call the polynomial \( P(X) = X^n + b_1X^{n-1} + \ldots + b_m \) \((b_\xi \in \mathbb{R})\) hyperbolic if all its roots are real.

**THEOREM 1.** Suppose that for any \( t \in (-1, 1) \) the polynomial \( P(t, X) = X^m + B_1(t)X^{m-1} + \ldots + B_m(t) \) is hyperbolic and the multiplicity of its roots does not exceed \( k \). We assume that the coefficients \( B_j(t) \) are \( k \) times continuously differentiable on the interval \((-1, 1)\). Then at any point \( t_0 \in (-1, 1) \) all the roots \( \lambda_j(t) \) \((j = 1, m)\) of the polynomial \( P(t, X) \) (with suitable choice of the branches) are differentiable.

**THEOREM 2.** Suppose that the polynomial \( P(t, y, X) = X^m + B_1(t, y)X^{m-1} + \ldots + B_m(t, y) \) is hyperbolic for any \((t, y) \in (-1, 1) \times \mathbb{R} \), where \( \mathbb{R} \) is a compact topological space, and that the multiplicity of its roots does not exceed \( k \). Suppose that \((\frac{\partial^j}{\partial t^j})B_j(t, y) \) \((i = 0, k; j = 1, m)\) are continuous functions on \((-1, 1) \times \mathbb{R}\). Then for any compact set \( K \subset (-1, 1) \times \mathbb{R} \) there is a constant \( C_K \) such that for the roots \( \lambda_j(t, y) \) of the polynomial \( P \) there is the estimate

\[ \left| \frac{\partial}{\partial t} \lambda_j(t, y) \right| < C_K \psi(t, y) \in K. \]

Remark. The second derivatives \((\frac{\partial^2}{\partial t^2})\lambda_j(t, y)\) may be unbounded even if the coefficients \( B_j \) are infinitely differentiable in \( t \). [Example: \( P(t, y, X) = X^2 + t^2 - y^2, y \in [-1, 1], (\frac{\partial^2}{\partial t^2})\lambda(t, y) = \pm y^2/(t^2 + y^2)^{3/2}. \)]

In the case where \( B_j(t, y) \) are analytic in \( t \), Theorem 2 is a simple corollary (cf. [1]) of Bochner's theorem on tubes [3, p. 197].

For what follows we need the following assertion (for the proof see, e.g., [4, p. 84]).

**LEMMA 1.** Let \( f(t, X) \) be continuous in \( t \) and holomorphic in \( X \) in a neighborhood of the point \((t_0, \lambda) \in \mathbb{R} \times \mathbb{C}\). If the equation \( f(t_0, \lambda) = 0 \) has an \( r \)-fold root \( \lambda \): \((\frac{\partial^i}{\partial X^i})f(t_0, \lambda) = 0 \) \((i = 0, r - 1)\), then in a sufficiently small neighborhood of the point \((t_0, \lambda) \in \mathbb{R} \times \mathbb{C}\) the equation \( f(t, X) = 0 \) has an \( r \)-fold root \( \lambda_j(t) \) (counting multiplicity)
and \( \lim_{t \to t_0} \lambda_j(t) = \lambda, j = 1, r. \)

**Proof of Theorem 1.** Let \( \lambda(t_0) \) be a \( q \)-fold root of the polynomial \( P(t_0, X) \). We consider the polynomial

\[ Q(t, X) = P(t, \lambda(t_0) + X) = \sum_{j=0}^{m-q} B_j(t) X^{m-j} + \sum_{j=0}^{q} A_j(t) X^{q-j}, \] (1)

where \( A_j(t) = (\frac{\partial}{\partial X})^{q-j} P(t, \lambda(t_0))/(q-j)! \)

We introduce the notation

\[ a_j^{(i)} = \left( \frac{d}{dt} \right)^i A_j(t_0). \] (2)

By hypothesis \( a_0^{(0)} \neq 0, a_j^{(0)} = 0 \) \((j = 1, \ldots, q)\).

We shall prove that

\[ a_j^{(i)} = 0, \quad (i = 0, j - 1; j = 1, q). \] (3)
(3) is equivalent to the fact that \( A_j \) is representable in the form 
\[ A_j(t) = (t - t_0)^{j_0} \tilde{A}_j(t), \]
where \( \tilde{A}_j(t) \) is a continuous function.

We suppose otherwise. Let \( j_0 \) be the smallest index for which (3) is not satisfied:
\[ a^{(i)}_j = 0 \quad (v_i < i_0), \quad a^{(i)}_j \neq 0 \quad (i_0 < j_0). \]
We make the change of variable \( X = (t - t_0) \tilde{X} \), where \( \tilde{X} = i_0/j_0 \). Then the hyperbolic polynomial \( (\partial X)^q \tilde{X} Q(t, X) \) (hyperbolicity is preserved in taking the derivative) after division by \( (t - t_0)^{j_0} \) can be written in the form
\[ a^{(i)}_0 q! \tilde{X}^i + \frac{(q - i)}{j_0!} a^{(i)}_0 \text{sign}(t - t_0)^i \tilde{X}^{m-q+i} + \ldots + \tilde{S}_m(t), \]
where \( \tilde{S}_i(t) \) are continuous functions with \( \tilde{S}_i(0) = 0 \). The equation \( a^{(i)}_0 \tilde{X}^{j_0} + a^{(i)}_0 \text{sign}(t - t_0)^{j_0} = 0 \) (0 \( < i_0 < j_0 \) \( \geq \) 2) has complex roots for \( (t - t_0) a^{(i)}_0 \text{sign}(t - t_0)^{j_0} > 0 \). If, moreover, \( |t - t_0| \) is sufficiently small, then according to Lemma 1 polynomial (4) also has a nonreal root contrary to the hypotheses of the theorem. The equality (3) has been proved.

It follows from (3) that with the change of variable \( X = (t - t_0) \tilde{X} \) Eq. (1) can be represented in the form
\[ (t - t_0)^{-q} Q(t, \tilde{X}(t - t_0)) = a^{(i)}_0 \tilde{X}^q + \frac{1}{q!} a^{(i)}_0 \tilde{X}^{q-1} + \ldots + \frac{1}{q!} a^{(i)}_0 + \tilde{S}_m(t) \tilde{X}^m + \ldots + \tilde{S}_m(t) = 0, \]
where \( \tilde{S}_i(t) \) are continuous functions with \( \tilde{S}_i(t_0) = 0 \).

According to Lemma 1, in a sufficiently small neighborhood of the point \( t_0 \) Eq. (5) has \( q \) roots (counting multiplicity) \( \tilde{\lambda}_1(t), \ldots, \tilde{\lambda}_q(t) \), which are continuous at the point \( t_0 \). Therefore, \( P(t, X) \) has \( q \) roots of the form \( \lambda_j(t) = \lambda(t_0) + (t - t_0) \tilde{\lambda}_j(t) \), i.e., all the roots taking the value \( \lambda(t_0) \) at the point \( t_0 \) are differentiable at this point, and \( \tilde{\lambda}_j(t_0) \) is found from the equation
\[ a^{(i)}_0 \tilde{X}^q + \frac{1}{q!} a^{(i)}_0 \tilde{X}^{q-1} + \ldots + \frac{1}{q!} a^{(i)}_0 = 0, \]
where
\[ \tilde{\lambda}_j(t_0) = \left( \frac{\partial}{\partial t} \right)^{q-j} P(t_0, \lambda(t_0)) \frac{1}{(q-j)!}. \]

The proof of Theorem 1 is complete.

We consider some properties of polynomials needed for the proof of Theorem 2.

**Lemma 2.** Suppose that the roots \( x_i \) of the polynomial
\[ P(X) = \sum_{j=0}^{m} b_j X^{m-j} = \prod_{i=1}^{m} (X - x_i), \quad b_i \in \mathbb{C}, \quad b_0 = 1 \]
satisfy the inequalities \( |x_1| \leq |x_2| \leq \ldots \leq |x_m| \). Then
\[ |x_1| < 2m^{\frac{3}{2}} \left( \min \left( \left| \frac{b_{m-1}}{b_{m-1}} \right|, \left| \frac{b_{m-2}}{b_{m-2}} \right|^{1/2} \right) \right) + \min \left( \left| \frac{b_{m-1}}{b_{m-1}} \right|, \left| \frac{b_{m-2}}{b_{m-2}} \right|^{1/2} \right)^{1/2}. \]

**Proof.** We consider two cases: a) \( |x_1| > |x_2|/2m \); b) \( |x_1| < |x_2|/2m \).

a) From Viet's formulas we have
\[ |x_1 x_2 x_3 \ldots x_m| = |b_m|, \quad |x_2 x_3 \ldots x_m| \geq |b_{m-1}|/m, \quad |x_3 \ldots x_m| \geq |b_{m-2}|/m^2. \]
Hence \( |x_1| \leq m |b_m/b_{m-1}| \) and \( x_1^2 \leq x_1 x_2 \leq m^2 |b_m/b_{m-2}|, \) whence \( |x_2| \leq 2m |x_1| \leq 2m^2 \min (|b_{m-1}|, |b_m/b_{m-2}|^{1/2}) \).

b) From Viet's formulas we have
\[ |x_2 \ldots x_m| = |1 + x_1 x_2 + x_1 x_3 + \ldots + x_1/x_m| = |b_m|, \]
and
\[ |x_2 \ldots x_m| \leq 2b_{m-1}, \]
\[ |x_3 \ldots x_m| \geq \left( \frac{m}{m-1} \right)^{1/2} |b_{m-2}|, \]
\[ |x_4 \ldots x_m| \geq \left( \frac{m}{m-1} \cdot \frac{m-2}{6} \right)^{1/2} |b_{m-3}|. \]

hence \( |x_2| \leq m^2 \min (|b_{m-1}|/b_{m-1}, |b_{m-1}/b_{m-3}|^{1/2}), \)

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