SOME METRIC PROPERTIES OF p-ADIC NUMBERS

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This article is devoted in the main to the transference to the p-adic number domain of the results of
the metric theory of continued fractions constructed by A. Ya. Khinchin for the real case, with some rein-
forcements and simplifications which are impossible in the real case.

A detailed exposition of all the necessary information on p-adic numbers can be found in [1-3].

Let us agree on the following convention: \( x = \sum_{i=n}^{\infty} a_ip^i \) is to be written in the form
\( a_{n+1}a_n + \ldots a_1a_0 \),
where the question is where to put the comma. We shall assume here that \( a_i = a_i(x) \) take only the values
\( 0, 1, 2, \ldots, p-1 \). The norm of a p-adic number is defined in this case by the relationship
\( \|x\| = p^{-k} \), where
\( k = \min \{ l : a_l \neq 0, -p \leq l \} \). We put \( \{x\} = a_{n+1}a_n + \ldots a_1a_0 \), \( [x] = 0, a_1a_2a_3 \ldots \). Let us denote by \( X \) the set
of all p-adic numbers of the form \( 0, a_1a_2a_3 \ldots \), the set of all p-adic integers which are not factors of unity,
and by \( Y \) the set of all p-adic numbers of the form \( a_{n+1}a_n + \ldots a_1a_0 \), \( 0 \ldots \). Obviously, \([x] \in X, \{x\} \in Y \). The
set \( X \) is a compact subgroup of a locally compact metrizable group of p-adic numbers relative to the opera-
tion of adding p-adic numbers and the topology induced in the field of p-adic numbers by the p-adic norm.
We shall denote by \( P \) the Haar measure of the additive group of p-adic numbers and we shall assume it
normed in such a way that \( P(X) = 1 \).

An arbitrary p-adic integer \( x \in X \) can be decomposed in a unique way into a continued fraction the ele-
ments of which will be p-adic numbers of the set \( Y \), just as each real number of the unit segment can in
a unique way be decomposed into a continued fraction with integer elements. We shall indicate the start
of this process. Let \( x \in X, x \neq 0 \). We put \( x_1(x) = \{1/x\} \). Then
\[ x = \frac{1}{x} = x_1(x) + \frac{1}{x_2(x) + \ldots} \]
If \( [1/x] \neq 0 \), then we proceed further in a similar manner. We write:
\[ x = \frac{1}{x_1(x)} \quad \frac{1}{x_2(x) + \frac{1}{x_3(x) + \ldots}} \]
For almost all \( x \in X \) this process can be continued indefinitely, since the set \( X \) itself possesses the power
of the continuum, and its subset consisting of p-adic integers admitting decomposition into a finite con-
tinued fraction of the type indicated is countable.

By analogy with the real case, we consider the convergents
\[ R_1(x) = \frac{1}{x_1(x)} ; \quad R_2(x) = \frac{1}{x_1(x) + \frac{1}{x_2(x)}} \text{ etc.} \]
We shall give a representation of some first convergents of a number in terms of elements of the continued
fraction which represents it:

The numerator and denominator of the convergent $R_n(x)$ in this representation we shall denote by $r_n(x)$ and $q_n(x)$, respectively. Direct checking shows that each $p$-adic integer $x \in X$ can be represented in a unique way by a continued fraction of the type indicated, and, conversely, that each such continued fraction represents some $p$-adic number $x \in X$ in the sense that the convergents converge to the number represented, and 

$$\|x - R_n(x)\| < p^{-n}, \quad \|R_n(x) - R_{n+1}(x)\| < p^{-n}. \quad \tag{1}$$

**Theorem 1.** $\|R_n(x) - R_{n+m}(x)\| \to 0$ is monotonic for arbitrary $x \in X$ given arbitrary natural $m$, i.e., if $N, n, k, l$ are natural numbers, $N > n$, then $\|R_N(x) - R_{N+k}(x)\| < \|R_N(x) - R_{N+l}(x)\|$.

This may be verified directly.

In the real case such a strong statement is, as we well know, simply untrue.

**Theorem 2.** $\|r_n(x)\| \geq p^{n-1}, \quad \|q_n(x)\| \geq p^n$ for all $x \in X$.

This may be proved by simple induction over $n$ (we note that $l\|x_i(x)\| \geq p$ for arbitrary $i$).

In the real case there is an analogous theorem stating that for continued fractions with integral elements $q_k \geq 2(k-1)/2 \quad (5$, Theorem 12).

**Theorem 3.** For arbitrary integral $i > 1$ and $k_j, 0 \leq k_j \leq p - 1 \quad (j = 1, 2, \ldots, i)$, the sets $\{x: a_i(x) = k_j\}$ in $X$ are independent relative to $P$ and $P\{x: a_i(x) = k_j\} = p^{-1}.$

The proof of this theorem follows directly from the invariance of the Haar measure.

**Theorem 4.** For arbitrary integral $i > 1$ and arbitrary $y_j \in Y \quad (j = 1, 2, \ldots, i)$ the sets $\{x: x_j(x) = y_j\}$ in $X$ are independent relative to $P$ and $P\{x: x_j(x) = y_j\} = p^{-2i}.$

Proof. Let $x = 0, a_1a_2a_3\ldots$ be such that $x_i(x)$ has a given value: $x_i(x) = a_{-n}, a_{-n+1}, a_{-n+2}, \ldots, a_{n-1}, a_n, a_{n+1}, a_{n+2}, \ldots, a_{2n}$ are unambiguously defined with respect to $a_{-n}, a_{-n+1}, a_{-n+2}, \ldots, a_{2n}$ are completely arbitrary. By Theorem 3, this gives for the measure the quantity $p^{-2i}$. It can be shown analogously that if two first elements of the decomposition $x_i(x)$ and $x_i(x)$ are given, and $\|x_i(x)\| = p^k, \quad \|x_i(x)\| = p^l$, then the measure of the corresponding set will be $p^{-2(k+l)}$. It is not difficult to see, on summing over all the $x_i(x)$, that if we had fixed only $x_2(x)$, and as before $\|x_i(x)\| = p^l$, we would have obtained the measure $p^{-2l}$. Thus we have proved the independence of the sets corresponding to $x_i(x)$ and $x_i(x)$. Independence can be proved in exactly the same way in the general case.

In the real case this does not hold, and only asymptotic independence holds (see [5], Theorem 34, or else [6]).

From this theorem we immediately have

**Theorem 5.** Let $C$ be an arbitrary positive number. Then

$$P\left\{ x: \|x_i(x)\| < C \right\} = 0. \quad \tag{2}$$

In the real case the position is analogous (see [5], Theorem 29).

For $p$-adic numbers a statement holds which is altogether analogous to Borel's theorem on normal numbers. We shall formulate and prove it for the set $X$. It is perfectly clear that the general statement follows directly from this.

**Theorem 6.** Almost all (relative to the measure $P$) $p$-adic numbers of the set $X$ contain in their decomposition the numbers $0, 1, 2, \ldots, p - 1$ with the same frequency, namely $1/p$.

Proof. Let us consider the following transformation $S$ of the set $X$ into itself:

$$0, a_1a_2a_3\ldots \rightarrow 0, a_2a_3a_4\ldots$$

This transformation is an ergodic transformation which conserves the measure. The fact that it conserves the measure is obvious, and the fact that it is ergodic can be shown, for example, in the following way.

$$R_1(x) = \frac{1}{x_1(x)}; \quad R_2(x) = \frac{x_2(x)}{x_1(x)x_2(x) + 1}; \quad R_3(x) = \frac{x_3(x)x_2(x) + 1}{x_1(x)x_2(x)x_3(x) + x_1(x)x_3(x) + x_3(x)}. \quad \tag{3}$$