ON ISOPERIMETRIC AND VARIOUS OTHER INEQUALITIES
FOR A MANIFOLD OF BOUNDED CURVATURE

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§ 1. FUNDAMENTAL DEFINITIONS AND FORMULATION OF RESULTS

Let $F$ be a compact two-dimensional manifold of bounded curvature $[1]$ with a nonempty edge $L$ consisting of a finite number of rectifiable curves, homeomorphic to a disc. We also denote by $F$ and $L$, respectively, the area of the manifold $F$ and the length of its boundary. Unless otherwise stated, the area or length of a set of the space (if it is meaningful to speak of them) will be denoted in the following by the same symbol as the set itself. This will not lead to confusion, since it will always be clear from the context what is meant.

We denote by $\omega(E)$ and $\chi(E)$, respectively, the curvature and the Euler characteristic of a set $E \subset F$.

The greatest distance from points $X \in F$ to $L$ will be called the internal radius of the space $F$, denoted by $R(F)$.

We shall define the excess of the curvature of $F$ with respect to a real number $\kappa$, as the difference

$$\omega_\kappa = \omega(F) - \kappa F.$$

The positive and negative parts $\omega_\kappa^+$ and $\omega_\kappa^-$ of the excess of the curvature $\omega_\kappa$ are defined by the equations

$$\omega_\kappa^+ = \sup_{E \subset F} \omega(E) - \kappa E, \quad \omega_\kappa^- = \inf_{E \subset F} \kappa E - \omega(E),$$

where $E$ are Borel sets. One can easily see (see [1], p. 236) that $\omega_\kappa^+$ and $\omega_\kappa^-$ are respectively equal to the positive and negative parts $\omega^+$ and $\omega^-$ of the curvature $\omega(F)$ if $\kappa = 0$.

In the formulations of various theorems of the present paper we make use of the function $\psi = \psi(x, a, b, K)$ ($x \geq 0, a \geq 0, b, K$ are real numbers), which satisfies the differential equation

$$\psi'' + \kappa \psi = b$$

and the initial conditions

$$\psi|_{x=0} = 0, \quad \psi'|_{x=0} = a.$$

In the following we shall also need the continuous single-valued function

$$x = x(\kappa, a, b, K),$$

defined by the following relations:

$$\psi(x(\kappa, a, b, K), a, b, K) = \psi, \quad x(0, a, b, K) = 0.$$

It is easy to obtain the following explicit expressions for the functions $\psi$ and $x$:

$$\psi = \begin{cases} \frac{b}{K} \cosh y - K x + \frac{a}{y^2} \sinh y - K x + \frac{b}{K} & \text{for } K < 0, \\ \frac{a x}{2} + \frac{b}{2} x^2 & \text{for } K = 0, \\ \frac{-b}{K} \cos y K x + \frac{a}{y^2} \sin y K x + \frac{b}{K} & \text{for } K > 0, \end{cases}$$

We now formulate the main results.

**THEOREM 1.** Let F(x) be the set of points of the manifold F which are not further from L than x. Then for any real number K and for any \( x \in [0, \min(R(F), \chi(K))] \)

\[
F(x) \leq \psi(x, L, \omega_x + KF - 2\pi\chi, K),
\]

where \( \chi = \chi(F), \sigma \)

\[
\chi(K) = \begin{cases} 
\frac{\pi}{2\sqrt{K}} & \text{for } K > 0, \\
\infty & \text{for } K \leq 0.
\end{cases}
\]

**THEOREM 2.** For any real number K the internal radius \( R(F) \) of F satisfies

\[
R(F) \geq \min[\chi(K), z(F, L, \omega_x + KF - 2\pi\chi, K)].
\]

**Remark 1.** Apparently,

\[
R(F) \geq z(F, L, \omega_x + KF - 2\pi\chi, K),
\]

although we are not yet able to prove this.

**Remark 2.** From Theorem 2 one can easily deduce the following theorem of Hadwiger (see [2], p. 47): if F is the area of a simply connected plane region and L is the length of its boundary, then there exists a circle of radius

\[
R(F) \geq 2F \\
L + \gamma L^2 - 4\pi F
\]

contained within this region.

The generalization of this theorem to the case of multiply connected plane regions (see [3]) also follows from Theorem 2.

**THEOREM 3.** (Isoperimetric Inequality.) If

\[
R(F) \leq \chi(K),
\]

then

\[
L^2 + 2(\omega_x - 2\pi\chi)F + KF^2 \geq 0.
\]

**Remark 1.** The condition (1) is only important if \( K > 0 \). Apparently, the inequality (2) is satisfied independently of the condition (1) for any real \( K \), although we are not yet able to prove this. Previously V. A. Toponogov has proved the inequality (2) without the restriction (1), but under the condition that

\[
\omega_x = 0, \chi = 1, K \geq 0.
\]

**Remark 2.** The isoperimetric inequality (2) is not, generally speaking, sharp inequality. However, it becomes a sharp inequality if certain additional restrictions are imposed on \( \omega_x, \chi, \text{ and } K \).

The inequality (2) is sharp, for example, if \( \omega_x < 2\pi \) and \( \chi = 1 \). Equality is attained in this case if F is a circle for which the specific curvature of any region not containing the center is equal to K and the curvature of the center is equal to \( \omega_x \).