The present paper arose from an attempt to understand the connection between the phenomenon of a cyclic object and the action of the circle on a topological space. The concept of cyclic object lies at the base of the construction of cyclic homology, introduced independently by A. Connes [1] and B. L. Tsygan [2]. There have already appeared dihedral, quaternionic, and symmetric objects [3, 4]. It turns out that all of them can be unified in the framework of the general concept of the action of a skew-simplicial group on a simplicial object.

It is shown in the first section that the category of skew-simplicial groups has a non-trivial final object \( W \), consisting of the Weyl groups of a system of roots of type \( B \). Each skew-simplicial group is an extension of some simplicial group by one of the seven objects in \( W \). The second section is devoted to connections with equivariant topology. It is proved that the geometric realization \( |G| \) of a skew-simplicial group \( G \) is a topological group. Moreover, the homotopy category of simplicial sets with action of \( G \) is equivalent to the homotopy category of topological \( |G| \)-spaces.

Let \( \Delta \) be the category of finite totally ordered sets \( [n] = \{0, 1, \ldots, n\} \) and non-decreasing maps. We consider the dual category \( \Delta^0 \). It is well known that its morphisms are generated by the simplest morphisms of the form

\[
d^i : [n] \to [n-1], \quad s^i : [n] \to [n+1], \quad 0 \leq i \leq n.
\]

which satisfy the following relations:

\[
d^i d^j = d^i-1 d^j, \quad i < j; \tag{1}
\]

\[
d^i s^j = \begin{cases} 
  s^j-1 d^i, & i < j, \\
  id, & i = j, \\
  s^j d^i, & i > j + 1; \\
  s^i s^j = s^i+1 s^j, & i \leq j. 
\end{cases} \tag{2}
\]

Definition 1.1. A small category \( \Sigma \) is called a category of type \( \Delta \) if it contains \( \Delta \) as a subcategory and has the same objects. In addition, it is required that each morphism \( f \in \text{Hom}_\Sigma ([n], [m]) \) can be represented uniquely as a composition \( f = \varphi \cdot g \) where \( g \in \text{Aut}_\Sigma [n] \) and \( \varphi \in \text{Hom}_\Delta ([n], [m]) \).

Thus, a category \( \Sigma \) of type \( \Delta \) is determined completely by a collection of automorphisms of \( \text{Aut}_\Sigma [n], n = 0, 1, \ldots \) and rules for commutation with the morphisms of \( \Delta \). The collection of such \( \Sigma \) forms a category \( \text{Cat} \Delta \) with functors preserving \( \Delta \) as morphisms.

Example 1.2. The category \( \Delta W \). We define a sequence of groups \( W_n = (Z/2)^{n+1} \times \Sigma_{n+1}, n = 0, 1, \ldots \)

\[
[n] \times \Sigma_{n+1} \to [n], \quad (i, \tau) \mapsto \tau^*(i),
\]

and \( Z/2 \) is the group with two elements \( \{+1, -1\} \). We define the multiplication

\[
((\varepsilon_0, \ldots, \varepsilon_n), \tau)((\eta_0, \ldots, \eta_n), \sigma) = ((\varepsilon_0 \cdot \eta_0, \ldots, \varepsilon_n \cdot \eta_{n+1}), \tau \cdot \sigma).
\]

On \( W \), we introduce the structure of a simplicial set, defining the face and degeneracy operators in the following way:
Finally, we define the category $\Delta W$ of type $\Delta$ by commutation relations in the dual form (i.e., for the category $\Delta W^0$):

$$d^i \omega = (d^i \omega) \cdot d^{\omega (0)}, \quad s^j \omega = (s^j \omega) \cdot s^{\omega (0)}. \quad (7)$$

Here $\omega^*$ denotes the action of the group $W_n$ on the set $[n]$, induced by the obvious projection $W_n \to \Sigma_n+1$.

**Definition 1.3.** A skew-simplicial group is a pair $(G, \gamma)$, consisting of a simplicial set $G$ and a simplicial map $\gamma: G \to W$, for which the following conditions hold:

1) the set $G_n$ is a group and the map $\gamma_n$ a homomorphism for each $n$;
2) for each $n$ and all $0 \leq i \leq n$, $0 \leq j \leq n$ one has

$$d^i (g_1, g_2) = (d^i g_1) \cdot d^i (\gamma (g_1) (i) g_2), \quad (8)$$

Here and below for awkward indices we use the abbreviations $d^i = d^i \gamma$, $s^j = s^j \gamma$.

By a morphism of skew-simplicial groups $(G, \gamma) \to (G', \gamma')$ is meant a simplicial map $f: G \to G'$ such that $f_n$ is a homomorphism for each $n$, while $\gamma = \gamma' \cdot f$. We denote the category of skew-simplicial groups by $\text{Skew-SGr}$.

**Theorem 1.4.** One has an isomorphism of categories

$$\text{Cat}_\Delta \cong \text{Skew-SGr}. \quad (9)$$

**Proof.** Let $\Sigma$ be a category of type $\Delta$. We construct a skew-simplicial group structure on $G = \text{Aut}_\Sigma \{.[.\}$ in four steps. For convenience we shall work in the dual category $\Sigma^0$. By the letter $g$ (possibly with dashes) we shall denote the elements of $G$.

**Step 1.** Structure of simplicial set. According to Definition 1.1, the morphism $d^i \cdot g$ of the category $\Sigma^0$ can be written uniquely in the form $g^* \cdot d^i$. The correspondence $g \mapsto g^*$ defines operators $d^i: G_n \to G_{n+1}$. Analogously, we get operators $s^j: G_n \to G_{n+1}$. The simplicial identities (1)-(3) automatically hold by virtue of the uniqueness.

**Step 2.** Construction of homomorphisms $e_n, \bar{e}_n: G_n \to \Sigma_{n+1}$. The equation $d^i \cdot g = g^* \cdot d^i$ from the first step also gives a correspondence $(d^i, g) \leftrightarrow g^*$, defining a right action of $G_n$ on the set $[n]$: $g^*(i) = j$. This is equivalent to giving a homomorphism $e_n: G_n \to \Sigma_{n+1}$ (cf. Example 1.2). In exactly the same way, from the equations $s^j \cdot g = g^* \cdot s^j$ we get a homomorphism $\bar{e}_n: G_n \to \Sigma_{n+1}$.

**Step 3.** The homomorphisms $e$ and $\bar{e}$ are equal. We introduce the abbreviations $g^* = e(g)^*$, $\bar{g}^* = \bar{e}(g)^*$. As a result of steps 1 and 2, we have the equations

$$d^i \cdot g = (d^i g) \cdot d^{\omega (0)}, \quad s^j \cdot g = (s^j g) \cdot s^{\omega (0)}, \quad (9)$$

which let us make the transformations

$$d^i \cdot s^j \cdot g = (d^i s^j g) \cdot d^{\omega (0)} (i) \cdot s^{\omega (0)} (j),$$

$$s^j \cdot d^i \cdot g = (s^j d^i g) \cdot d^{\omega (0)} (j) \cdot s^{\omega (0)} (i).$$