COEFFICIENT PROBLEM FOR UNIVALENT FUNCTIONS WITH QUASICONFORMAL EXTENSION

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1. Introduction

The analytic functions with quasiconformal extension occupy a somewhat special position in the geometric theory of functions. On one hand, they can be considered as a subclass of the whole class of univalent functions in a given domain and thus, the obtained distortion estimates give a significant refinement of the general distortion theorems for univalent functions. On the other hand, the properties of the analytic functions with a quasiconformal extension are important in other areas: for example, they are essential in the theory of Teichmüller spaces and its applications.

However, there exist problems which are identical in formulation to the problems of the general theory of univalent analytic functions, but their solution requires completely different methods. One of these interesting and unsolved problems in the theory of analytic functions with a quasiconformal extension remains the determination of sharp estimates for the Taylor coefficients. The corresponding problem for the class of all normalized univalent functions in the circle forms the content of the Bieberbach conjecture, formulated almost 70 years ago and solved only recently in a brilliant manner by de Branges [1].

The purpose of the present paper is to establish sharp coefficient estimates for univalent functions with (normalized) k-quasiconformal extensions for small k. This answers a question of Kuhnau and Niske in [2], and solves the coefficient problem for mappings with small deviations. In addition, we intend to discuss here the question of the uniform estimation of all the coefficients for a given k.

In addition to the inner properties of conformal and quasiconformal mappings, the proofs will make use of a technique, similar to the one applied in the theory of Teichmüller spaces, as well as (implicitly) of some properties of the Caratheodory metric.

The results of this paper have been communicated in [3].

2. Classes S(k). The Kuhnau-Niske Problem

We consider the known class S of analytic functions f(z) univalent in the unit circle

\[ \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \]

and with expansion

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1) \]

along with its subclasses S(k)(0 \leq k < 1), formed by functions of the form (1), admitting k'-quasiconformal extensions \( \hat{f} \) to the whole of Riemann sphere \( \hat{\mathbb{C}} \) with \( k' \leq k \) and with the additional normalization \( \hat{f}(\infty) = \infty \). The closure of the union \( \bigcup_k S(k) \) in the topology of uniform convergence on compact subsets of \( \Delta \) coincides with S.
Let \( \mu_I(z) = \frac{\partial f}{\partial z} \) be the Beltrami coefficient (or, in other words, the complex characteristic) of a mapping \( f = \mu \in S(k) \), and let \( \Delta^* = \{ z \in C : |z| > 1 \} \) and
\[
B(\Delta^*) = \{ \mu \in L_\infty(C) : \mu|_\Delta = 0, \| \mu \|_\infty < 1 \}.
\]
Clearly that all \( \mu_I \in B(\Delta^*) \). For small \( \| \mu \|_\infty \leq k \) for the mappings \( f = \mu \in S(k) \) one can make use of the variation formula
\[
f^\mu(z) = z - \frac{2}{\pi} \int_{\Delta^*} \frac{\mu(z) \, dx \, dy}{\overline{z}^2 (z - \xi)},
\]
from where there follows that
\[
a_n = -\frac{1}{\pi} \int_{\Delta^*} \frac{\mu(z) \, dx \, dy}{|z|^{n+1}} + O(\| \mu \|_\infty), \quad n = 2, 3, \ldots,
\]
from where one obtains at once the estimate
\[
|a_n| \leq \frac{2k}{n-1} + O(k^2), \quad k \to 0.
\]
This estimate was obtained as far back as [4] and was later established by other methods in [5, 6]. It is asymptotically sharp in the sense that the factor \( 2/(n-1) \) in front of \( k \) cannot be decreased; the ratio \( O(k^2)/k^2 \) is uniformly bounded for all \( k \leq k_0 < 1 \). Indeed, as shown in [7], the estimate (3) admits the following refinement, valid for all \( k \) \((0 < k < 1)\):
\[
|a_n| \leq \frac{2k}{n-1} + \left( n + \frac{2}{n-1} \right) k^2.
\]
In [2] one has formulated the following question: does there exist (for each fixed \( n \geq 3 \)) a constant \( k_n > 0 \) such that for \( 0 < k \leq k_n \) one has the inequality
\[
|a_n| \leq \frac{2k}{n-1} \quad \text{or for each } k, 0 < k < 1, \text{ is there an } f \in S(k) \text{ with } |a_n(f)| > 2k/(n-1) ?
\]
It is well known that for \( n = 2 \) the estimate (5) holds:
\[
|a_n| = \frac{2k}{n-1} \quad \text{for all } k < 1,
\]
and the equality sign prevails only for the functions
\[
f_l(z) = \begin{cases} \frac{z}{|z|^{n-1}} = z + \sum_{n=2}^{\infty} nt_n z^n, & |z| \leq 1, \\ \frac{|z|^2}{(z^{1/2} - t z^{1/2})}, & |z| \geq 1, \end{cases}
\]
where \( t = ke^{i\theta}, 0 \leq \theta \leq 2\pi \). For a fixed \( n \geq 3 \) the equality \( |a_n| = \frac{2k}{n-1} \) is satisfied for the mappings
\[
f_{l, n-1}(z) = (f_l(z^{n-1}))^{1/(n-1)},
\]
for which the expansion in the circle \( \Delta \) has the form
\[
f_{l, n-1}(z) = z + \frac{2t}{n-1} z^n + a_{2n-1} z^{2n-1} + \ldots,
\]
while the Beltrami coefficient in \( \Delta^* \) is
\[
\mu_{l, n-1}(z) = t \mu_n(z), \quad \mu_n(z) = -|z|^{n+1}/z^{n+1}.
\]
At the solving of the indicated problem in [8] with the aid of Grunsky's inequality for the functions \( \sqrt{f(z^2)} \) it is established that for even \( n = 2m \) the remainder in the right-hand side of (3) can be estimated by a quantity \( O_m(k^2) \).

In [7, 9], the author, with the aid of completely different arguments, has established that for small \( k \) for the extremal function \( f_0 \) realizing \( \max |a_n| \) on \( S(k) \) we have for all \( p \neq n \), \( l \) the relation
\[
a_{p,l}(f_0) = O_p(k^2), \quad k \to 0,
\]