We study tolerance control with a control device which registers all instances of a process $X(t)$ exceeding the normal state zone $[c, d]$, where $d - c = l$ (see Fig. 1). The following assumptions are adopted: the quantity being controlled is a differentiable time function with normal probability distribution density

$$f(X) = \frac{1}{\sqrt{2\pi} \sigma_X} \exp \left\{ -\frac{(X - m_X)^2}{2\sigma_X^2} \right\},$$

where $m_X$ and $\sigma_X$ are the expectation and the standard deviation of the process $X(t)$, respectively.

The reliability of the control of $X(t)$ for increasing the sampling interval will naturally decrease. We determine the relationship between sampling intervals of the variable observed and the specified value of the criterion characterizing the reliability of control. To solve this problem, we make use of certain aspects of the overshoot theory [1].

Suppose that at time point $t_n$ a control operation has been executed, which established whether the value of the quantity monitored is within the prespecified interval $[c, d]$. If it is so, we assume that until the next control operation it will remain at the same level with a certain probability. For the criterion of reliability of control results, one can then take the probability $p\{X \in [c, d]\}$ that at $t_n < t < t_{n+1}$ the value monitored will remain within the tolerance interval boundaries.

We begin by estimating the probability that within the infinitesimal time interval $\delta$ that follows directly the time point $t_n$, the process will step outside the tolerance interval $[c, d]$. For $X(t)$ to cross the boundary $X = d$ in these conditions two events are necessary: first, at the time point $t_n$ the value of the quantity monitored should be less than $d$, i.e., $X(t_n) < d$. Second, at the time point $t_n + \delta$ the value of $X$ should be greater than $d$, i.e.,

$$X(t_n+\delta) > d.$$  \hfill (1)

Therefore, the probability that process $X(t)$ will step beyond line $X = d$ within time interval $\delta$ can be written as

$$p[X(t_n) < d; X(t_n+\delta) > d].$$ \hfill (2)

Likewise, for the probability that process $X(t)$ will exceed the limit of $X = c$ in interval $\delta$, we write

$$p[X(t_n) > c; X(t_n+\delta) < c].$$ \hfill (3)

Proceeding from the differentiability of a random function $X(t)$ on a short time interval $\delta$ we can express $X(t_n+\delta)$ as

$$X(t_n+\delta) \approx X(t_n) + \dot{X}(t_n) \delta,$$ \hfill (4)

where $\dot{X}(t_n)$ is the derivative of the quantity monitored at the time point $t_n$ when the preceding control operation was executed.

We see from (4) that (1) is equivalent to the inequality

$$d - \dot{X}(t_n) \delta < X(t_n).$$
Instead of two inequalities which characterized (2) the event of the excess of the process monitored beyond the boundary of the straight plot \( X = d \) in the interval \( \delta \), we can now write a single dual inequality:

\[
d - \dot{X}(t_n) \delta < X(t_n) < d \quad (\dot{X}(t_n) > 0).
\]

Likewise, the passage of \( X(t) \) beyond \( X = c \) in the interval \( \delta \) is expressed (see Fig. 1) by the inequality

\[
c < X(t_n) < c - \dot{X}(t_n) \delta \quad (\dot{X}(t_n) < 0).
\]

For the probabilities sought we have

\[
p(d - \dot{X}(t_n) \delta < X(t_n) < d) = \int_{d - \dot{X}_0}^{d} \int_{d - \dot{X}_0}^{d} f(X, \dot{X}) dX d\dot{X},
\]

\[
p(c < X(t_n) < c - \dot{X}(t_n) \delta) = \int_{c - \dot{X}_0}^{c} \int_{c - \dot{X}_0}^{c} f(X, \dot{X}) dX d\dot{X},
\]

where the integration limits cover all values of \( X(t_n) \) and \( \dot{X}(t_n) \), which satisfy inequalities (5) and (6), respectively.

The integration limits of the inner integrals in (7) differ by an infinitesimal quantity \( \dot{X}\delta \). Therefore, with the aid of the mean value theorem we obtain

\[
\int_{d - \dot{X}_0}^{d} \int_{d - \dot{X}_0}^{d} f(X, \dot{X}) dX d\dot{X} = \delta \int_{d - \dot{X}_0}^{d} f(d, \dot{X}) \dot{X} d\dot{X},
\]

\[
\int_{c - \dot{X}_0}^{c} \int_{c - \dot{X}_0}^{c} f(X, \dot{X}) dX d\dot{X} = -\delta \int_{c - \dot{X}_0}^{c} f(c, \dot{X}) \dot{X} d\dot{X},
\]

where \( f(d, \dot{X}) \) and \( f(c, \dot{X}) \) are probability densities of the system of random quantities \( X \) and \( \dot{X} \) taken at arguments \( X = d \) and \( X = c \), respectively.

Substituting (8) into (7) we have

\[
p(d - \dot{X}(t_n) \delta < X(t_n) < d) = \delta \int_{d - \dot{X}_0}^{d} f(d, \dot{X}) \dot{X} d\dot{X},
\]

\[
p(c < X(t_n) < c - \dot{X}(t_n) \delta) = -\delta \int_{c - \dot{X}_0}^{c} f(c, \dot{X}) \dot{X} d\dot{X}.
\]

The probability that process \( X(t) \) is in the tolerance interval \([c, d]\) during an infinitesimal time interval is now written as

\[
p^* = |1 - p[d - \dot{X}(t_n) \delta < X(t_n) < d]| \cdot (1 - p[c < X(t_n) < c - \dot{X}(t_n) \delta]) = 1 + \delta \int_{-\infty}^{0} f(c, \dot{X}) \dot{X} d\dot{X} - \delta \int_{0}^{\infty} f(d, \dot{X}) \dot{X} d\dot{X} - \delta \int_{-\infty}^{0} f(d, \dot{X}) \dot{X} d\dot{X} + \delta \int_{0}^{\infty} f(c, \dot{X}) \dot{X} d\dot{X}.
\]