ALGORITHMS FOR DIGITAL MEASUREMENT OF RANDOM-SIGNAL ROLLING MEAN AND VARIANCE

M. Ya. Mints and V. N. Chinkov

Recurrent algorithms have been derived for calculating the rolling mean and variance, which reduce the memory volume required, improve the speed, and raise the reliability.

Hardware analysis of random-signal characteristics is constantly extending and so it is important to develop methods and means of measurement [1, 2]. In particular, real-time process monitoring and control require estimation of the current or rolling mean and current variance for random signals, which is due to the increase in the classical algorithms result in more complicated hardware and slower calculations.

For example, there are classical algorithms for calculating the estimators for the mean $\bar{X}_N$ and biased variance $D_N$ of a set of $N$ elements:

$$\bar{X}_N = \frac{1}{N} \sum_{k=1}^{N} x_k,$$

$$D_N = \frac{1}{N} \sum_{k=1}^{N} (x_k - \bar{X}_N)^2,$$

in which $x_k$ is element $k$ in the set, $k = 1, N$. This constitutes the code for the instantaneous value of the random signal $x(t)$, i.e., $x_1 = x(t_1)$, which is derived by analog-digital conversion.

We can write (2) as

$$D_N = \frac{1}{N} \sum_{k=1}^{N} x_k^2 - (\bar{X}_N)^2.$$

There are certain difficulties in the direct use of algorithms for calculating (1), (2), and (3) when $N$ is large. In hardware implementation of (2), one requires an executive store to hold the $N$ numbers $x_k$, which may complicate the device and reduce the reliability if $N$ is large. Algorithm (3) requires an increase in the number of bits in all the memory cells and elevated computational accuracy because the determination of $D_N$ requires one to calculate the sums $\sum_{k=1}^{N} x_k$ and $\sum_{k}^{N} x_k$ for each current value of $N$ from a large number of terms, so the sums for large $N$ are much larger than the individual terms, and when one derives $D_N$ as the difference of two terms, rounding is impermissible, since the information on the value of $D_N$ is contained in the least-significant bits.

We propose recurrent algorithms for calculating the current mean and variance, which on the one hand provide real-time information on the process dynamics and on the other reduce the volume of the statistic processed at each step, which simplifies the processor structure.

The rolling-mean algorithm is obtained directly from (1) as
\[ \bar{X}_N = \frac{x_N}{N} + \frac{N-1}{N} \bar{X}_{N-1}, \]

in which \( \bar{X}_{N-1} \) is the estimator for the mean from the sample volume \( N-1 \).

The recurrent relation (4) enables one to derive \( \bar{X}_N \) from the previous value \( \bar{X}_{N-1} \) when the next sample element \( x_N \) comes in.

We derive a recurrent relation for the rolling variance estimator. For definiteness, we use the expression for the unbiased estimator for the sample variance in the form

\[ \sigma^2 = \frac{1}{N} \sum_{k=1}^{N} (x_k - \bar{X}_N)^2, \]

in which

\[ \Delta x_N = \bar{X}_N - \bar{X}_{N-1} = \frac{(x_N - \bar{X}_{N-1})}{N}, \]

so

\[ x_N - \bar{X}_{N-1} = N \Delta x_N. \]

We transform (5) to

\[ D_N = \frac{1}{N} \sum_{k=1}^{N-1} (x_k - \bar{X}_{N-1})^2 - \frac{1}{N} (x_N - \bar{X}_{N-1})^2 - (\Delta x_N)^2 \]

We substitute (3) into this expression along with

\[ \sum_{k=1}^{N-1} (x_k - \bar{X}_{N-1})^2 = (N-1)D_{N-1}, \]

to get

\[ D_N = \frac{N-1}{N} D_{N-1} + (N-1)(\Delta x_N)^2. \]

Formula (7) represents the recurrent algorithm for calculating the estimator for the current variance \( D_N \) from the previous value \( D_{N-1} \) and the current value of \( \Delta x_N \). We transfer to the unbiased variance estimator \( D'_N = (N/(N-1))D_N \) and get

\[ D'_N = \frac{N-2}{N-1} D'_{N-1} + N(\Delta x_N)^2, \]

in which \( D'_N \) and \( D'_{N-1} \) are the unbiased variance estimators based on \( N \) and \( N-1 \) set elements.

In some cases, it is necessary to obtain estimators for the mean and variance not only for the entire observation time but also in parallel for certain time intervals, each of which contains a certain group of set elements. This can be done by extending the above algorithms.

Let there be two groups of elements: \( \{x_{1i}\}, i = 1, N, \) and \( \{x_{2k}\}, k = 1, N_2, \) where \( N_1 \) and \( N_2 \) are the volumes of the first and second groups.

The following define the resultant estimator for the mean \( \bar{X}_N \) and bias estimator for the variance \( D_N \) in the combined group of \( \{x_{1i}\} \) and \( \{x_{2k}\} \):

\[ \bar{X}_N = \frac{1}{N} \left( \sum_{i=1}^{N_1} x_{1i} + \sum_{k=1}^{N_2} x_{2k} \right); \]

\[ D_N = \frac{1}{N} \left[ \sum_{i=1}^{N_1} (x_{1i} - \bar{X}_N)^2 + \sum_{k=1}^{N_2} (x_{2k} - \bar{X}_N)^2 \right], \]