APPLICATION OF THE METHOD OF MAXIMAL COMPACTNESS FOR FORECASTING CHANGES IN METROLOGICAL CHARACTERISTICS OF STANDARDS

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By means of the method of maximal compactness, the problem of structural and parametric identification of autoregressive drift-models of metrological characteristics of working standards is solved.

The physical cause of metrological failures of working standards (WS) is the irreversible changes in their metrological characteristics (MCH) as the bases of the elements age. These changes are usually slow and of a monotone nature. It is therefore well known that forecasting of results of comparisons of WS is directly associated with determination of systematic component and estimation of a random component of the drift of MCH of WS under uncertain conditions due to the errors in the reference measurements. However, in such a seemingly simple measurement problem which is solved under laboratory conditions, already in the course of identification of the model of the systematic component of the drift of MCH of WS based on comparative data a number of difficulties arise:

- the results of WS comparisons constitute short series;
- the structure of models of the drift of MCH of WS is not determined due to insufficient duration of experimental investigations [1] and incomplete control of operational factors;
- prior information about probabilistic characteristics of the errors in reference measurements is not available.

Thus, identification of drift models of MCH of WS represents a typical initial problem of mathematical statistics [2] and its correct solution can be obtained by means of the method of maximal compactness (MMC).

Since the initial information for forecasting the results of the next comparison of a WS involves solely the results of proceeding comparisons in the form of time series, we shall use adaptive models [3, 4] as the forecasting tool for MCH of a WS. These are based on the hypothesis that the process under investigation is an output signal of a linear filter at whose input a white noise acts and the terms of the time series $x_t$ at time $t$ are weighted sums of the current and previous values of the input stream

$$x_t = \mu + \varepsilon_t + \psi_1 x_{t-1} + \psi_2 x_{t-2} + \ldots + \psi_l y_t,$$

where $\mu = \text{const}$ is the location characteristic of the process; $\varepsilon_t$ is the white noise with variance $\sigma_\varepsilon^2$; $B$ is the backward shift operator such that

$$B x_t = x_{t-1}, \quad B^m x_t = x_{t-m}, \quad \psi(B) = \psi_1 B + \psi_2 B^2 + \ldots.$$

If the sequence $\psi_1, \psi_2, \ldots$ converges, the filter is stable, the process $x_t$ is stationary and $\mu$ is the mean value of the process. Otherwise the process $x_t$ is not stationary and $\mu$ has no special meaning except for being a reference point of the process' level [5].

Autoregressive model is the simplest model obtained by means of a linear filtering of white noise. In it the current value of the process is expressed in terms of a linear set of the preceding values and the disturbance $\varepsilon$. Denoting the deviation from the mean by $x_t = x_t - \mu$, we arrive at the process

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TABLE 1. Errors of Forecasting of the Drift of MCH of a D1-13 Apparatus by Means of Autoregressive Models

<table>
<thead>
<tr>
<th>n</th>
<th>m=0</th>
<th>m=1</th>
<th>result of forecasting</th>
<th>forecasting error</th>
<th>result of forecasting</th>
<th>forecasting error</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td></td>
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<tr>
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<tr>
<td>6</td>
<td>4</td>
<td>2</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

\[ x_t = \varphi_1 x_{t-1} + \varphi_2 x_{t-2} + \cdots + \varphi_p x_{t-p} + \varepsilon_t, \]

where \( \varphi_i = \text{const}, \ i = 1, p. \) This process is called autoregressive process of order \( p \) and is denoted by AR(\( p \)). If one introduces an autoregressive operator \( \varphi(B) = 1 - \varphi_1 B - \varphi_2 B^2 - \cdots - \varphi_p B^p \) of order \( p \) then the model becomes of the form \( \varphi(B) x_t = \varepsilon_t \). In this model there are \( p+2 \) unknown parameters \( \mu, \varphi_1, \varphi_2, \ldots, \varphi_p, \sigma^2 \), which ought to be estimated based on the available data about the process under investigation.

A model of another type — which is of practical value — is the model of finite moving averages (MA) in which \( x_t \) depends linearly on a finite number of preceding values of \( \varepsilon \), i.e.

\[ \dot{x}_t = \varepsilon_t - \theta_1 \dot{x}_{t-1} - \theta_2 \dot{x}_{t-2} - \cdots - \theta_q \dot{x}_{t-q}. \]

This is a moving averages process of order \( q \) or briefly MA(q). The weight coefficients \( 1, -\theta_1, -\theta_2, \ldots, -\theta_q \) are not necessarily positive or add up to 1.

If we introduce an operator of a MA(q) process

\[ Q(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q, \]

then the model MA(q) will be written in the form \( x_t = Q(B) \varepsilon_t \). The model involves \( q+2 \) unknown parameters \( \mu, \theta_1, \theta_2, \ldots, \theta_q, \sigma^2 \).

To achieve larger flexibility in construction of a model of the processes under investigation, it is desirable to include in it both the moving average and the autoregressive terms. This results in the mixed model ARMA (p, q) \( \varphi(B) x_t = Q(B) \varepsilon_t \) with \( p+q+2 \) unknown parameters.

The model can also be written as

\[ x_t = \theta_1 (B) Q(B) \varepsilon_t = \frac{Q(B)}{\theta(B)} \varepsilon_t = \frac{1 - \theta_1 B - \cdots - \theta_q B^q}{1 - \varphi_1 B - \cdots - \varphi_p B^p} \cdot \varepsilon_t. \]

One can also utilize the idea [6] of transforming nonstationary series into stationary ones by proceeding from the initial series to its differences of order \( d \). Here we introduce a generalized, autoregressive operator \( \Psi(B) x_t = \varphi(B) (1-B)^d \) where \( \varphi(B) \) is the stationary operator described above. Such a model can be represented as follows: \( \Psi(B) x_t = \varphi(B) (1-B)^d x_t = Q(B) \varepsilon_t \) or \( \varphi(B) w_t = Q(B) e_t \), where \( w_t = \nabla^d x_t, \nabla \) is the backward difference operator \( \nabla x_t = x_t - x_{t-1} = (1-B)x_t \).

This model is called integrated autoregressive moving averages (IARMA) model of order (p, d, q).

The model can be slightly modified by adding a constant term on the r.h.s. This new term generalizes the model to

\[ \Psi(B) x_t = \varphi(B) \nabla^d x_t - Q_0 + Q(B) \varepsilon_t. \]