Minimum Dominating Cycles in 2-Trees

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We consider the class of 2-trees and present a linear time algorithm for finding minimum dominating cycles of such graphs. We stress the use of a particular representation of these graphs called a recursive representation, and some linear operations on directed trees associated with these graphs.

KEY WORDS: Graph theory; algorithm; 2-tree domination; Hamiltonian cycle.

1. INTRODUCTION

The literature on the subject of domination is steadily growing. A recent survey of it may be found in Ref. 4. Algorithmically, the problem of finding a minimum dominating set in an arbitrary graph is known to be NP-complete. However, a linear algorithm exists for finding a minimum dominating set in a tree. This result has been extended to find R-bases of trees. Dominating circuits of a graph have been defined. In this paper we construct a minimum dominating cycle of a 2-tree if such a cycle exists, and establish the nonexistence of a dominating cycle in the opposite case. We concentrate our attention on maximal outerplanar graphs, for which our algorithms were originally designed. In addition to this work, an algorithmic study of maximal outerplanar graphs was initiated.

An outerplanar graph is a graph that can be embedded in a plane in such a way that every vertex lies on the exterior face. A maximal outerplanar graph (hereafter called a mop) is an outerplanar graph such that the addition of an edge between any two nonadjacent vertices results in a graph that is not outerplanar. There are several characterizations of mops, two of which are of interest here. The first is that a graph is a mop iff it is isomorphic
to a triangulation of a polygon. The second is that a graph is a mop iff it can be constructed from a (base) triangle by a finite number of applications of the following operation: to the graph already constructed add a new vertex in the exterior face and join it to two adjacent vertices on the exterior face. Figure 1a illustrates a mop constructed by this process starting from a base triangle labeled 1, 2, 3. The $i$th vertex for $i > 3$ is joined to two vertices with labels less than $i$. A generalization of this construction gives us a notion of 2-trees. For this class of graphs, the operation of recursive addition, as described above for mops, is not limited to vertices on the exterior face of the existing graph. The new vertex may be joined to any two adjacent vertices (or a $K_2$ subgraph—hence the name) of the existing graph.

With any mop we can associate a tree \(\text{(associated tree)}^{(1)}\) or dual tree\(^{(1)}\) which is obtained by placing a node inside each triangle of the mop and joining two nodes iff the corresponding triangles have an edge in common (see Fig. 1). In this paper we investigate the idea that certain properties of a mop can be determined by examining its associated tree. In particular we are interested in algorithms for solving certain routing problems in mops. For example, what is the length of a shortest closed walk (route) that comes within distance one of every vertex in a mop. Stated more formally, a cycle $C$ in a graph $G$ is a dominating cycle if every vertex not in $C$ is adjacent to at least one vertex in $C$. A minimum dominating cycle in $G$ has minimum length among all dominating cycles in $G$. In what follows we use the associated tree of a mop $G$ to construct a linear algorithm for finding a minimum dominating cycle in $G$.

Not all 2-trees have dominating cycles. We establish a sufficient and necessary condition for existence of a dominating cycle of a 2-tree. The

![Fig. 1. A mop (a), its associated tree (b), and the recursive representation (c).](image-url)