BASIS PROBLEM FOR A FREE ALTERNATIVE $\Phi$-ALGEBRA

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Despite the fact that in recent years many interesting results pertaining to the structure of a free alternative $\Phi$-algebra have been obtained [1–4], there is still no effective method for constructing a basis. In the present paper we construct a basis for a $\Phi$-module whose elements are those of a free alternative $\Phi$-algebra with three generators, linear in one of the generators, over an integral domain $\Phi$.

1. Recall that a $\Phi$-algebra is called alternative if it satisfies the identities

$$(xx)y = x(xy), \ (xy)x = x(yx).$$

(1)

Suppose $x, y, z$ are elements of an alternative $\Phi$-algebra. We will employ the following notation: $[x, y, z] = (xy)z - x(yz)$ is the associator of $x, y, z$; $[x, y] = xy - yx$ is the commutator of $x, y$; $[xy]^n = \prod_{i=1}^{n} \{x, y\}$; $[xy]^m = \prod_{i=1}^{m} \{x, y\}$; $R_x$ is right multiplication by $x$; $L_x$ is left multiplication by $x$; $T_x$ is right or left multiplication by $x$.

In the multiplication ring of an alternative $\Phi$-algebra we have the relations

$$R_x = R_y + L_y R_x - R_x L_y,$$

(2)

$$L_x = L_y + R_y L_x - L_x R_y.$$  

(3)

We will also use the following well-known identities:

$$(x, xy, z) = (x, y, z)x,$$

(4)

$$(x, yz, z) = x(x, y, z),$$

(5)

$$(x, y, z) : xy = (x, y, z)y \cdot x,$$

(6)

$$xy \cdot (x, y, z) = y \cdot x(x, y, z).$$

(7)

Proofs of these identities can be found, e.g., in [5, 6]. Furthermore, we will use the identities

$$\sum_{i=0}^{n} (-1)^i a^i x a^{n-i} = [xa^n]$$

(8)

(here and later on, $a^0$ denotes the empty word) and

Proof of (8) can be found in [7], and (9) is given in [8].

Suppose \( \mathfrak{A} \) is an integral domain of characteristic different from 2, and \( R \) is a free alternative algebra with three generators \( a, b, c \) over \( \Phi \). We introduce the following notation: \( \mathfrak{A} = \mathfrak{A}/(c)^2 \), \( K_1 = (c, a, b) \), \( K_r = (K_{r-1}, a, b) \), where \( r \geq 2 \); \( D_n \) is the ideal of algebra \( \mathfrak{A} \), generated by the element \( K_n \). Clearly, \( D_1 = D_2 = \ldots = D_n \supseteq \ldots \). Note that \( D_1 \) is the associator ideal of \( \mathfrak{A} \). In the sequel we will denote it by \( D \). Also, for any \( x, y \in D \) we will write \( x \equiv y \pmod{D_n} \). Clearly, \( \equiv \) is an equivalence relation on \( D \).

Since the quotient algebra \( \mathfrak{A}/D \) is associative, any element of \( \mathfrak{A} \) can be represented as a sum of elements with a right arrangement of parentheses and elements of the ideal \( D \). The aim of the present paper is to show that the set \( B = \{b_i(K_r a^i b^j)\} \) is a basis of the ideal \( D \) as a \( \Phi \)-module.

2°. In this section we will show that any element of \( D \) is a linear combination of elements of the set \( B \).

Lemma 1. For any \( x \in D \) we have the equalities \( x(ab) = (xb)a \), \( x(ba) = (xa)b \), \( (ab)x = b(ax) \), \( (ba)x = a(bx) \).

Proof. Identities (2) and (3) enable us to represent any element \( x \in D \) as a linear combination of elements of the form \( K_r T_y \ldots T_{y_n} \), where \( y_i \in (a, b) \). If we first apply identities (4) and (5), then (6) and (7), and then again (4) and (5), we obtain \( x = x_1 + x_2 + \ldots + x_{r+1} \). This implies the first of the equalities being proved. The others are handled analogously.

Lemma 2. For any \( x \in D \), we have

\[
x R_b R_a = x R_a R_b, \quad x R_a L_b = x L_a R_b, \quad x L_b R_a = x R_b L_a, \quad x R_b L_a = x L_b R_a.
\]

Proof. By Lemma 1, \( x R_b R_a = (xb)a = x(ab) \equiv (xa)b = x R_a R_b \). The remaining cases are handled analogously.

Lemma 3. The set \( B = \{b_i(K_r a^i b^j)\} \) is a system of generators of the \( \Phi \)-module \( D \).

Proof. In view of Lemma 2, it is clear that any element of the ideal \( D \) is equivalent \( \pmod{D_{r+1}} \) to a linear combination of elements of the form \( b_i(K_r a^i b^j) \). An element of this form equivalent to \( z \) will be denoted by \( z_r \). Then for each \( x \in D \) there exist elements \( x^{(i)} \in D \) such that \( x = x_1 + x_2 + \ldots + x_{r+1} \). Since the number of generators appearing outside the associators in the elements \( x^{(i)} \) decreases with each step, there exists an \( m \) such that \( x^{(m)} = 0 \). Adding \( m-1 \) equalities termwise, we obtain \( x = x_1 + x_2 + \ldots + x_{m-1} \). Since the elements appearing on the right-hand side have the prescribed form, the theorem is proved.

3°. In this section we will prove that the set \( B \) is linearly independent over \( \Phi \).

Lemma 4. For any natural number \( r \), any nonnegative integer \( m \), and any nonzero element \( \alpha \in \Phi \) we have

\[
\alpha [K_r a^m b^n] \neq 0.
\]

Proof. Let \( \mathbf{K} \) be a Cayley algebra over \( \mathbf{Z} \). Since relation (10) holds in the algebra \( \Phi \otimes \mathbf{K} \), it is also true in \( \mathfrak{A} \), but its left-hand side is linear relative to \( c \), hence relation (10) holds in \( \mathfrak{A} \).

Lemma 5. For any natural numbers \( m, n, r \) and any nonzero \( \alpha \in \Phi \) we have

\[
\alpha [K_r a^m b^n] \neq 0.
\]

Proof. If \( \alpha [K_r a^m b^n] = 0 \), then

\[
\alpha [K_r a^m b^n] = \sum_{y_1 \ldots y_n} K_{r+1} T_{y_1} \ldots T_{y_n},
\]

where \( y_1, \ldots, y_n \in \{a, b\} \). Since \( b \) is a free generator, it follows that

\[
\alpha [K_r a^m (a + b)^n] = \sum_{z_1 \ldots z_n} K_{r+1} T_{z_1} \ldots T_{z_n},
\]

where \( z_1, \ldots, z_n \in \{a, b\} \). Each term on the right-hand side has degree greater than \( r \) with respect to \( b \). On the left-hand side there is a single term \( [K_r a^{rn+p}] \) having degree \( r \) with respect to \( b \). In view of the homogeneity