Transient Heat Transfer Between a Perfect Conductor with Heat Generated in it and a Semi-infinite Solid Including a Contact Resistance

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Abstract. Analytical solutions are presented for heat conduction between a semi-infinite solid and a perfect conductor with internal heat generation, taking into account a thermal contact resistance at the interface.

1 Description of the Problem

The boundary and initial conditions are:
- Heat is supplied to the perfect conductor at constant rate Q/M per unit mass and unit time.
- The initial temperature of the solid is zero.
- The initial temperature of the perfect conductor is \( u \).
- There is heat transfer between the perfect conductor and the surface of the solid \( (x = 0) \) at rate \( H \) times their temperature difference.

We write \( c' \) for the specific heat of the perfect conductor, \( u \) for its temperature and \( M \) for the mass in contact with unit area of the surface \( (x = 0) \) of the solid.

The temperature of the solid is \( v \), its thermal conductivity \( K \) and its thermal diffusivity \( \kappa \).

Then the heat conduction problem can be described by the equations

1. Energy conservation in the perfect conductor

\[
Q = Mc' \frac{du}{dt} + H(u - v) \quad |x = 0, t > 0
\]

2. Energy equation in the semi-infinite solid

\[
\frac{d^2v}{dx^2} - \frac{1}{\kappa} \frac{dv}{dt} \quad |x > 0, t > 0
\]
3. Initial conditions
\[ v = 0 \quad | \quad t = 0 \]  
\[ u = V \quad | \quad t = 0 \]  

4. Boundary condition
\[ H(u - v) + K \frac{dv}{dx} = 0 \quad | \quad x = 0. \]  

2 Solution of the Problem

When the temperature of the perfect conductor is assumed to be a constant \( u = V \), the solution for the temperature field can be obtained from the literature e.g. Carslaw & Jaeger [1]

\[ v = V \cdot \left[ \text{erfc} \left( \frac{x}{2\sqrt{\kappa t}} \right) - e^{hx h^2 \kappa t} \cdot \text{erfc} \left( \frac{x}{2\sqrt{\kappa t}} + h'i(t-t') \right) \right] \]  

with
\[ h = \frac{H}{K}. \]  

Following Duhamel's theorem, the temperature \( v \) in the case of time-dependent \( u(t) \) is given by the convolution integral

\[ v = \int_{0}^{t} u(t') \frac{\partial}{\partial t'} \left[ \text{erfc} \left( \frac{x}{2\sqrt{\kappa (t-t')}} \right) - e^{hx h^2 \kappa (t-t')} \cdot \text{erfc} \left( \frac{x}{2\sqrt{\kappa (t-t')}} + h'i(t-t') \right) \right] dt'. \]  

and, at \( x = 0 \)

\[ v(0,t) = \int_{0}^{t} u(t') \frac{\partial}{\partial t'} \left[ 1 - e^{hx h^2 \kappa (t-t')} \cdot \text{erfc} \left( h'i(t-t') \right) \right] dt'. \]  

Introducing
\[ \varepsilon = \frac{Q}{HV} \]  
\[ \mu = \frac{M \cdot c'}{H} \cdot h^2 \kappa \]  
and writing dimensionless temperatures
\[ u^+ = \frac{u}{V} \]  
\[ v^+ = \frac{v}{V}, \]  

the dimensionless time \( \tau = h^2 \kappa t \) and the dimensionless position \( \xi = h \cdot x \) one obtains from Eq. (1)

\[ \varepsilon = \mu \frac{\partial u^+}{\partial \tau} + (u^+ - v^+) \bigg|_{\xi=0}. \]  

From the Laplace transform of Eq. (16) one gets with respect to the initial condition 4

\[ \frac{\xi}{t} = \mu (u^+ \cdot p - 1) + (\hat{u}^+ - \hat{v}^+) \bigg|_{\xi=0}. \]  

Transformation of Eq. (8) leads to

\[ v^+ = \frac{u^+}{\sqrt{p + 1}} \cdot \frac{1}{\sqrt{p}} \cdot e^{-\sqrt{p} \cdot \xi}. \]  

Substituting Eq. (18) into Eq. (17) one obtains

\[ \hat{u}^+ = \frac{\sqrt{p} + 1}{\sqrt{p}(p + \sqrt{p} + \mu - 1)} \left( \frac{\xi}{p \mu} + 1 \right) \]  

\[ \hat{v}^+ = \frac{1}{\sqrt{p}(p + \sqrt{p} + \mu - 1)} \left( \frac{\xi}{p \mu} + 1 \right) e^{-\sqrt{p} \xi}. \]  

By defining the constants \( a \) and \( b \) by the Equation

\[ \frac{1}{p + \sqrt{p} + \mu - 1} = \frac{1}{\sqrt{p} + a} \frac{1}{\sqrt{p} + b} \]  

the inverse Laplace transformation of the Eqs. (19) and (20) leads to the temperatures \( u^+ \) and \( v^+ \) [1, 2]

\[ v^+ = e^{-\frac{\xi^2}{4\tau}} \left[ W \left( \sqrt{\tau} \cdot (\xi + \mu) + \frac{\xi}{2\sqrt{\tau}} \right) \right] \cdot \varepsilon \]  

\[ + \frac{\xi}{a-b} \left[ \frac{a}{b} W \left( \sqrt{\tau} \cdot b + \frac{\xi}{2\sqrt{\tau}} \right) - \frac{b}{a} W \left( \sqrt{\tau} \cdot a + \frac{\xi}{2\sqrt{\tau}} \right) \right] \]  

where the function \( W \) is defined as

\[ W(z) = e^{z^2} \text{erfc} z. \]  

At \( \xi = 0 \) one obtains for the temperature of the solid

\[ v^+ = \left( \frac{2}{\sqrt{a}} \sqrt{\tau} \cdot \mu + \frac{\xi}{a-b} \left[ \frac{a}{b} W (\sqrt{\tau} \cdot b) - \frac{b}{a} W (\sqrt{\tau} \cdot a) \right] \right) \]  

\[ + \frac{1}{a-b} \left[ W (\sqrt{\tau} \cdot b) - W (\sqrt{\tau} \cdot a) \right] \bigg|_{\xi=0}. \]