DETERMINATION OF THE POTENTIAL ENERGY OF A GRAVITATIONAL SYSTEM WITH SPHEROIDAL SYMMETRY

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In this paper we generalize Ambartsumyan's "strip" method to systems with spheroidal symmetry. We obtain equations which can be used to compute the mass-luminosity ratio for two elliptic galaxies NGC 4406 \((f/f_G = 46)\) and NGC 4486 \((f/f_G = 48)\). The estimates given in the paper are in satisfactory agreement with the results of other authors.

The potential energy of a gravitational system can be found if we know the volume distribution of the material density. If the system is spherically symmetric, the volume density can be found from an observable in the plane of the figure by solving Abel's integral equation. In [1], Ambartsumyan proposed a simple method for determining the potential energy of a spherically symmetrical system. He showed that if the volume density function is projected onto an arbitrary axis, the potential energy of the system is proportional to the integral of the square of the one-dimensional density distribution function (the strip function) over the whole region.

The aim of this paper is to generalize Ambartsumyan's method to systems with spheroidal symmetry.

1. By a system with spheroidal symmetry we mean a system for which the surfaces of equal density are similar and similarly situated spheroids of the family defined by the equation

\[ e^2 (x'^2 + u'^2) - z'^2 = e^2 k^2, \]

where \(e\) is the true sphericity of the system and the plane \(z' = 0\) is the equatorial plane. The volume density distribution function for such a system can be written as a function of the parameter \(k\) in generalized spherical coordinates defined by the equations

\[
\begin{align*}
x' &= k \cos \hat{\theta}_1 \sin \theta_2, \\
y' &= k \sin \hat{\theta}_1 \cos \theta_2, \\
z' &= ke \cos \theta_2.
\end{align*}
\]

We project the density distribution function \(\rho(k)\) on the plane of the sky. If the angle between the line of sight and the axis of symmetry of the system is \(\hat{\iota}\) and \(x\) and \(y\) are Cartesian coordinates in the plane of the sky, the observable density distribution function is defined by the equation

\[
P(x, y) = 2 \rho(x) \int_{\rho(x)}^{\infty} \frac{dz}{\left(\frac{x^2 + a^2(xy)}{a(x)\sqrt{xy}}\right)^{3/2}},
\]

where

\[ \varphi_0 (x, y) = \sqrt{x^2 + \frac{y^2}{e^2}}. \]

and

\[ \varepsilon = \sqrt{\frac{e^2 \sin^2 \theta + \cos^2 \theta}{\varepsilon^2}} \]

is the apparent sphericity of the system. Kholopov derived an equation similar to (3) in [2]. We note that the x axis is directed along the major semi-axis of the system, while the y axis is directed along the minor semi-axis.

The one-dimensional density distribution function along the minor semi-axis of the system is found by integrating \( P(x, y) \) from \( x = -\infty \) to \( x = +\infty \):

\[ \varphi (y) = \int_{-\infty}^{\infty} P(x, y) \, dx. \]

(5)

If we change the variable of integration in (5)

\[ x = \sqrt{k^2 - \frac{y^2}{\varepsilon^2} \cos \omega} \quad \text{and} \quad \sqrt{\frac{\sigma^2}{\varepsilon^2} (x, y) = \sqrt{k^2 - \frac{y^2}{\varepsilon^2} \sin \omega}} \]

and integrate with respect to \( k \) from \( y/\varepsilon \) to \( \infty \) and with respect to \( \omega \) from 0 to \( \pi/2 \), we obtain

\[ \varphi (y) = 2\pi \int_{y/\varepsilon}^{\infty} \varphi (k) \, dk. \]

(6)

Similarly we can obtain an equation for the density distribution function along the x axis

\[ \varphi (x) = 2\pi \int_{y/\varepsilon}^{\infty} \varphi (k) \, dk. \]

(7)

We introduce \( \Phi (y/\varepsilon) = \varphi (y) \); then from (6) there follows

\[ \frac{d}{dx} \Phi \left( \frac{y}{\varepsilon} \right) = -2\pi \frac{e}{\varepsilon} \varphi \left( \frac{y}{\varepsilon} \right) \frac{y}{\varepsilon}. \]

(8)

Similarly, we obtain an equation for \( \psi (x) \)

\[ \frac{d}{dx} \psi (x) = -2\pi \frac{e}{\varepsilon} \varphi (x) x. \]

(9)

2. To compute the potential energy of the system we select a thin layer \( E_k \) in the volume bounded by two similar spheroids of semi-axes \( k, \varepsilon k \) and \( (k + \Delta k), (k + \Delta k)\varepsilon \). Then the element of volume of mass \( dm' \) at the point \( P' \) inside the layer \( E_k \) has potential energy (which we write with reversed sign)

\[ dU' = V (P') \, dm', \]

(10)

where \( V (P') \) is the potential due to the layer at the internal point \( P' \). By Newton's theorem that the potential is constant inside a homogeneous ellipsoidal layer [3], we have