Haar Measures on Hopf $C^*$-Algebras

XIU-CHI QUAN*
Institute of Fundamental Theory, University of Florida, Gainesville, FL 32611, U.S.A.

(Received: 26 August 1992)

Abstract. In this paper, we first use Markov–Kakutani's fixed point theorem to prove the existence and uniqueness of Haar measures on cocommutative Hopf $C^*$-algebras. Also we show that in the commutative case, there exists a natural one-to-one correspondence between the Haar measure on a given Hopf $C^*$-algebra and Haar measures on the associated semigroup. Finally, we show that for Hopf $C^*$-algebras with Peter-Weyl property, they have Haar measures.


Key words. Haar measure, Hopf $C^*$-algebra, fixed points, quantum group.

1. Introduction

The existence and uniqueness of Haar measures on a compact group is a fundamental result in the study of compact groups. Compact quantum groups are the subjects of active research [5, 6]. They are interesting in mathematics, and are closely related to the monoidal category, knot theory, and they have wide range applications on mathematical physics. It is natural to ask: Does there exist Haar measures on compact quantum groups? If yes, is it unique?

In [8], Woronowicz used representations of compact matrix pseudo-groups to show the existence and uniqueness of Haar measures on compact matrix pseudo-groups, this approach can only apply to limited classes of Hopf $C^*$-algebras. In the study of compact groups, we use Haar measures to study the representations of compact groups. Woronowicz [8] considered the special case of matrix groups (in the dual Hopf $C^*$-category) and proved existence and uniqueness (up to scale) in this case. He first constructed representations, and then derived the Haar measure from this. Our result is more general and we go back to the fundamental properties of Hopf $C^*$-algebras. In this paper, we will provide a new approach to studying Haar measure on Hopf $C^*$-algebras, we show that for cocommutative Hopf $C^*$-algebras and Hopf $C^*$-algebras with Peter–Weyl property, there exists a unique Haar measure. We expect that Haar measures will play an important role in the study of representations of Hopf $C^*$-algebras, the subject of a forthcoming paper.

The contents of the present paper are organized as follows: In Section 2, we first show that the states of a Hopf $C^*$-algebra with unit form a compact convex semigroup; then by using the Markov–Kakutani fixed point theorem, we show the

*Work supported in part by the NSF.
existence and uniqueness of Haar measures on cocommutative Hopf C*-algebras with unit. In Section 3, by using Riesz's representation theorem, we establish a natural one-to-one correspondence between the Haar measure on commutative Hopf C*-algebras and the Haar measure on locally compact semigroups. As a consequence of this, we show that for any compact Abelian semigroup with unit, there exists Haar measure on it. In Section 4, we show that for any involutive Hopf C*-algebra with Peter-Weyl property, there exists a unique Haar measure.

2. Haar Measure on Cocommutative Hopf C*-Algebras

In this section, we will prove the existence and uniqueness of a Haar measure on a cocommutative Hopf C*-algebra with unit; this class of Hopf C*-algebras obviously includes the class of group Hopf C*-algebras associated with compact groups.

Let us begin the section by recalling the definition of Hopf C*-algebras. Throughout this section, we always assume that every Hopf C*-algebra has unit, and the C*-homomorphisms are unital.

**Definition 2.1.** Let \( A \) be a C*-algebra, \( \Phi: A \to A \otimes A \) a C*-homomorphism. We say that \((A, \Phi)\) is a Hopf C*-algebra if the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{\Phi} & A \otimes A \\
\phi \downarrow & & \downarrow \Phi \otimes \text{id} \\
A \otimes A & \xrightarrow{\text{id} \otimes \phi} & A \otimes A \otimes A
\end{array}
\]

\( \Phi \) is called the comultiplication of \( A \). \((A, \Phi)\) is said to be cocommutative if \( \tau \circ \Phi = \Phi \), where \( \tau: A \otimes A \to A \otimes A \) is the flip automorphism, \( \tau(a \otimes b) = b \otimes a \), \( \forall a, b \in A \). If \( k: A \to A \) is a *-anti-isomorphism of period 2, we say that \((A, \Phi, k)\) is an involutive Hopf C*-algebra if the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{k} & A \\
\phi \downarrow & & \uparrow \tau \\
A \otimes A & \xrightarrow{k \otimes k} & A \otimes A
\end{array}
\]

If \( e: A \to \mathbb{C} \) is a nonzero C*-homomorphism, we say that \( e \) is a counit of \((A, \Phi)\) if the following diagram commutes

\[
\begin{array}{ccc}
A \otimes \mathbb{C} & \xrightarrow{\sim} & A & \xrightarrow{\sim} & \mathbb{C} \otimes A \\
\text{id} \otimes e & & \phi & & e \otimes \text{id} \\
A \otimes A & \xrightarrow{\phi} & A \otimes A
\end{array}
\]

If \((A, \Phi, k, e)\) is an involutive Hopf C*-algebra with counit \( e \), such that the following diagram commutes