A Precise Upper Limit
for the Correctness of the Navier—Stokes Theory with Respect to the Kinetic Theory

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The condition of positive normal pressures must hold for all solutions in the kinetic theory, but is violated by the Navier—Stokes equations for sufficiently high distortions. A dimensionless measure of this discrepancy is furnished by the tension number. In order for all pressures to be positive, it is necessary and sufficient that the tension number be less than 1. If this condition is violated, the normal-stress effects of the kinetic and Navier—Stokes theories are of opposite sign.

**KEY WORDS:** kinetic theory of gases; Navier—Stokes theory of fluids; tension number; truncation number; Knudsen number; normal-stress effects.

It is well known that according to the kinetic theory of monatomic, moderately rarefied gases, the Navier—Stokes constitutive equation is only an approximation. While various arguments have been put forward to this effect, they all rest on pictorial remarks or on purely formal processes of approximation to solutions of the Maxwell—Boltzmann equation which belong to a hypothecated special class, the existence of which remains still a matter of conjecture. Even more than this, the arguments labor under confusion of the sufficient with the necessary. Typically, the author presents some calculation intended to convince the reader that certain assumptions in the kinetic theory reduce it to the Navier—Stokes theory, whereupon he proclaims that these assumptions must hold in order for the Navier—Stokes theory to follow!

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2 An exception is the work of Maxwell. First, Maxwell makes no such converse claim. Second, from his Eqs. (121) and (75), a necessary and sufficient condition could be formulated, but it would...
In fact, a simple necessary condition for the Navier–Stokes theory can be derived by rigorous, though trivial, mathematics, directly from the definitions and without use of the Maxwell–Boltzmann integrodifferential equation, let alone of any special process of integration of it. The key to the argument is that while the older studies always focused upon the shear stress, modern continuum mechanics has taught us to expect that departures from classical predictions manifest themselves first in normal-stress effects (see, for example, Truesdell and Noll\cite{2}). The crux of the argument is that while the definition of the pressure tensor in the kinetic theory forces the normal pressure on every element of area in every flow to be positive\(^3\) unless the mass density vanishes, the Navier–Stokes theory makes a tension result on elements which are suffering sufficiently violent distortion. When these two normal-stress effects are contrary, the underlying theories cannot square with each other.

Indeed, we know from the definitions\(^4\) that the Navier–Stokes theory can never follow from the kinetic theory except subject to the Stokes viscosity relation. Thus, for comparison, we need consider only the case when the pressure tensor \(P\) is given in terms of the stretching tensor \(D\) as follows:\(^4\):

\[
P = (p + \frac{\mu}{2} \text{Tr} \, D)I - 2\mu D
\]

where \(D_0\) is the deviator of \(D\), often called the distortion tensor, and where \(p\) and \(\mu\), the pressure and the shear viscosity, are positive functions of density and temperature.\(^5\) Hence,

\[
n \cdot Pn = p - 2\mu n \cdot D_0n
\]

Therefore, the normal pressure on the element normal to \(n\) is positive if and only if

\[
2n \cdot D_0n < p/\mu
\]

be very complicated and would not have any obvious interpretation either in terms of the mean free path, etc., or in terms of measurable quantities. [These two equations are exact, but Eq. (121) is valid only for Maxwellian molecules.] Likewise, in the voluminous papers of Boltzmann I have found no claim that any particular condition of the gas is a necessary one for the theory of linear viscosity to hold.

\(^3\) While I cannot find this conclusion in Chapman and Cowling,\(^5\) it follows immediately from Eq. (2) of Section 2.31. We must distinguish here between the true kinetic theory and the Chapman–Enskog iterates, the status of which has never been made precise except in very special cases. The condition of positive normal pressures holds for all exact solutions. If the Chapman–Enskog process converges to a particular exact solution, then this condition holds in the limit, but it certainly does not hold automatically at each finite stage. For example, while it does hold at the zeroth stage, which corresponds to Eulerian hydrodynamics, it does not hold automatically at the first stage, which corresponds to the Navier–Stokes theory. Presumably, the results at that stage are of value only insofar as they approximate properties of certain exact solutions. Hence arises the problem solved in the text, to which the preceding sentences of this footnote could serve as an alternative introduction.

\(^4\) This tensor and the other kinematical apparatus used here are defined and described by Truesdell and Toupin,\(^5\) for example. See Sections 82 and 83.

\(^5\) In fact, \(p = kn\theta\), where \(n\) is the number density and \(\theta\) is the temperature, but we do not need this explicit formula until we come to the special case below [Eq. (10)].