Optimal Bounds for Conduction in Two-Dimensional, Multiphase, Polycrystalline Media

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A complete geometrical characterization is given of the set of all possible effective conductivity tensors for two-dimensional composites made of an arbitrary number of given anisotropic phases. It is rigorously established that any polycrystalline composite formed from an arbitrary (possibly infinite) number of phases can be replaced by a composite formed from only two of the given phases without altering the effective conductivity tensor.

KEY WORDS: Polycrystalline composites; optimal bounds; effective conductivity; homogenization.

1. INTRODUCTION

This paper is concerned with the determination of bounds on the effective conductivity of a multiphase polycrystalline material with arbitrary phase geometry. Historically, attention was first restricted to composites with isotropic phases: Hashin\(^1\) gives a review of this early work. The subsequent developments for polycrystalline media\(^2\) focused on overall bounds on the individual eigenvalues of the effective conductivity tensor and no attempt was made to correlate these eigenvalues.

In a two-dimensional setting, a characterization of the set of all anisotropic conducting materials resulting from the mixture in arbitrary volume fraction of two anisotropic conducting phases was first proposed by Lurie and Cherkaev\(^3\). A more complete characterization was obtained by Francfort and Murat\(^4\) based on the ideas of H-convergence and compensated compactness developed by Murat and Tartar\(^5\-10\).
Here we obtain a complete characterization of the set of all possible effective conductivity tensors for two-dimensional composites made of an arbitrary number of given anisotropic phases. The eigenvalues of the conductivity tensors of the phases are assumed to be bounded above and below. Our main result, as anticipated by Lurie and Cherkaev, is that any polycrystalline composite formed from an arbitrary (possibly infinite) number of phases can be replaced by a composite formed from only two of the given phases without altering the effective conductivity tensor.

Lamination is a straightforward process of generating composites from one or two anisotropic phases. Specifically, each material is sliced orthogonal to one of its principal conductivity axes. It is then layered with the same material sliced in the orthogonal direction or with the other material sliced in either direction.

Let \((\alpha_1, \alpha_2)\) and \((\beta_1, \beta_2)\) be the respective eigenvalue pairs of the conductivity tensors of phase 1 and phase 2 and assume, with no loss of generality, that

\[
\alpha_1 \leq \alpha_2, \quad \beta_1 \leq \beta_2, \quad \alpha_1 \alpha_2 \leq \beta_1 \beta_2 \tag{1.1}
\]

A straightforward calculation\(^{(4)}\) establishes that the eigenvalues \(\lambda_1\) and \(\lambda_2\) of the effective conductivity tensors obtained through the aforementioned lamination processes lie on one of the following curves:

\[
\lambda_1 \lambda_2 = \alpha_1 \alpha_2, \quad \text{with} \quad \alpha_1 \leq \lambda_1, \lambda_2 \leq \alpha_2 \tag{1.2}
\]

\[
\lambda_1 \lambda_2 = \beta_1 \beta_2, \quad \text{with} \quad \beta_1 \leq \lambda_1, \lambda_2 \leq \beta_2 \tag{1.3}
\]

\[
\lambda_1 \text{ (or } \lambda_2) = [(\beta_1 - \alpha_1) \lambda_1 \lambda_2 + (\beta_2 - \alpha_2) \alpha_1 \beta_1] / (\beta_1 \beta_2 - \alpha_1 \alpha_2)
\]

\[
\text{with } \alpha_1 \alpha_2 \leq \lambda_1 \lambda_2 \leq \beta_1 \beta_2 \tag{1.4}
\]

\[
\lambda_1 \text{ (or } \lambda_2) = [(\beta_2 - \alpha_2) \lambda_1 \lambda_2 + (\beta_1 - \alpha_1) \alpha_2 \beta_2] / (\beta_1 \beta_2 - \alpha_1 \alpha_2)
\]

\[
\text{with } \alpha_1 \alpha_2 \leq \lambda_1 \lambda_2 \leq \beta_1 \beta_2 \tag{1.5}
\]

or on one of the curves obtained by interchanging the subscripts 1 and 2 on \(\alpha\) in (1.4) and (1.5).

It is rigorously proved by Francfort and Murat\(^{(4)}\) that an arbitrary conductivity tensor with eigenvalues \(\lambda_1\) and \(\lambda_2\) can be obtained as an effective tensor for the mixture of the two original phases if and only if the eigenvalue pair \((\lambda_1, \lambda_2)\) lies inside the outermost region of the \((\lambda_1, \lambda_2)\) plane bounded by the curves (1.2)–(1.5). Three cases have to be distinguished:

Case 1: \(\alpha_1 \leq \beta_1\) and \(\alpha_2 \leq \beta_2\)

Case 2: \(\alpha_1 < \beta_1\) and \(\alpha_2 > \beta_2\)

Case 3: \(\alpha_1 > \beta_1\) and \(\alpha_2 < \beta_2\)