Time Evolution of a One-Dimensional Point System:
A Note on Fritz's Paper

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In a companion paper (Ref. 5) Fritz studies the time evolution of a one-dimensional point system which was introduced by Spitzer as a model of traffic. In the present paper we improve some of the results of Ref. 5 by using a different approach. Our results are obtained in a very simple and straightforward way but the techniques employed require conditions somewhat stronger than those assumed in Ref. 5.

KEY WORDS: Nonlinear diffusion equations; hydrodynamical behavior; coupling of random walks.

There is a very simple and direct way to derive and improve some of the results of Ref. 5, as we shall see in this paper. Our method, as compared to that of Ref. 5, requires stronger assumptions and our techniques seem inadequate for studying extensions to many-dimensional cases. On the other hand our procedure looks very simple and more flexible so it might become useful in the analysis of other models.

We consider the equations

\[
\frac{d}{dt} \delta(n, t) = U'(\delta(n + 1, t)) + U'(\delta(n - 1, t)) - 2U''(\delta(n, t)) \quad (1a)
\]

\[
\delta(n, 0) = \delta(n) \quad (1b)
\]

where \( n \in \mathbb{Z}, t \geq 0, U \in C^\infty(\mathbb{R}) \), and it is a convex symmetric function of \( r \in \mathbb{R} \). Throughout we shall consider the case when there are positive constants \( a, b, \delta', \delta'' \) so that

\[
0 < a \leq U''(r) \leq b < \infty \quad (2)
\]

\[
0 < \delta' \leq \delta(n) \leq \delta'' < \infty \quad (3)
\]

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We refer to Ref. 5 for a discussion on the interpretation and meaning of Eq. (1). It is easy to see that Eq. (1) has a unique solution and that 
\[ \delta(n, t) = \delta, \] for all \( n \in \mathbb{Z} \) and all \( t \geq 0 \), is a solution of Eq. (1a), i.e., \( \delta(n) = \delta \) is a stationary profile. It is less obvious but still true that these are the only stationary profiles; see the Remarks to Lemma 2 below.

The question we are interested in concerns the time evolution of "slowly varying" initial profiles, i.e., the hydrodynamical behavior of Eq. (1); cf. Ref. 5 for the use of such terminology and Ref. 3 for a survey on the subject. We state precisely what is the problem and its answer in the following.

**Theorem 1.** Assume \( U \in C^\infty(\mathbb{R}) \) is a symmetric and convex function and that Eq. (2) holds. Let \( \varepsilon \in (0, 1] \) and denote by \( \delta \varepsilon(n, 0) = \Delta(\varepsilon n) \), where \( \Delta(r) \) is in \( C^\infty(\mathbb{R}) \) and

\[
0 < \delta' \leq \Delta(r) \leq \delta'' < \infty, \quad \|\Delta'\| = \sup_{r \in \mathbb{R}} |\Delta'(r)| \leq \delta''' < \infty
\] (4)

for suitable \( \delta', \delta'', \delta''' \) and where \( \Delta' \) denotes the derivative of \( \Delta \). Define for \( \varepsilon^{-1} r \in \mathbb{Z} \) and \( \tau \geq 0 \)

\[
\Delta \varepsilon(r, \tau) = \delta \varepsilon (\varepsilon^{-1} r, \varepsilon^{-2} \tau)
\] (5)

and let \( \Delta \varepsilon(r, \tau) \) for \( r \in \mathbb{R} \) and \( \tau \geq 0 \) be its linear interpolation. Then

\[
\lim_{\varepsilon \to 0} \Delta \varepsilon(r, \tau) = \Delta(r, \tau)
\] (6)

where \( \Delta(r, \tau) \) is in \( C^\infty(\mathbb{R} \times \mathbb{R}^+) \) and it satisfies the equation

\[
\frac{\partial}{\partial \tau} \Delta(r, \tau) = \frac{\partial}{\partial r} \left[ U''(\Delta(r, \tau)) \frac{\partial}{\partial r} \Delta(r, \tau) \right]
\] (7a)

\[
\Delta(r, 0) = \Delta(r)
\] (7b)

The proof of Theorem 1 is obtained in two steps: first we prove that \( \Delta \varepsilon(r, \tau) \) is an equicontinuous and bounded family of functions, so that by the Ascoli–Arzelà theorem (4) it converges by subsequences on the compacts. In the second step we prove that the limiting points satisfy Eq. (7), hence they coincide.

**Equicontinuity.** The function \( U'(\delta): \mathbb{R}^+ \to \mathbb{R}^+ \) is invertible so we can change variables and go from \( \delta(n, t) \) to \( u(n, t) \), where

\[
u(n, t) = U'(\delta(n, t))
\] (8)