On Mean Field Equations for Spin Glasses

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In this paper we study rigorously the random Ising model on a Cayley tree in the limit of infinite coordination number $z \to \infty$. An iterative scheme is developed relating mean magnetizations and mean square magnetizations of successive shells far removed from the surface of the lattice. In this way we obtain local properties of the model in the (thermodynamic) limit of an infinite number of shells. When the coupling constants are independent Gaussian random variables the SK expressions emerge as stable fixed points of our scheme and provide a valid local mean-field theory of spin glasses in which negative local entropy (at low temperatures) while perfectly possible mathematically may still perhaps be physically undesirable. Finally we examine the TAP equations and show that if the average over bond disorder and the limit $z \to \infty$ are actually performed, one recovers our iterative scheme and hence the SK equations also in the thermodynamic limit.

KEY WORDS: Cayley tree; iteration; fixed point; spin glass; Gaussian distribution; local mean-field theory; SK equations; TAP equations.

1. INTRODUCTION

Since the original work of Edwards and Anderson\textsuperscript{(1)} and Sherrington and Kirkpatrick (SK),\textsuperscript{(2)} much has been written about the validity of these authors’ mean-field theories for spin glasses (Refs. 3–5 and references quoted therein).

The random Curie–Weiss model considered by SK and the resulting mean-field-type equations have a certain appeal to them but, as pointed out by SK, these equations lead to negative entropy at low temperatures. Use of the so-called $n$-replica trick and interchanges of limits have been suggested as causes of this unphysical phenomenon and various theories have

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been proposed to overcome these difficulties. It is probably fair to say, however, that there is still no universally acceptable mean-field theory of spin glasses at this time, nor are there any rigorous results on the range of validity of the SK equations.

An alternative approach to the problem was proposed by Thouless, Anderson, and Palmer (TAP)\(^6\) in which the \(n\)-replica trick was avoided by taking thermal averages at the outset on a lattice with large coordination number \(z\). The resulting TAP equations apparently lead to a physically acceptable entropy but there are still difficulties in performing the average over bond disorder and controlling the mean-field limit \(z \to \infty\).

In their derivation, TAP argue that for large \(z\) the only terms in a graphical expansion that are important are those corresponding to a Bethe lattice or Cayley tree. On the other hand, Katsura, Inawashiro, and Fujiki\(^7\) consider a variant of the Bethe approximation, and also without use of the replica trick, obtain the SK equations in the limit \(z \to \infty\).

In order to understand this apparent discrepancy we present here a rigorous analysis of the random Ising model on a Cayley tree in the limit of infinite coordination number \(z\). What we obtain in this limit is an iterative scheme

\[
\begin{align*}
m_{i+1} &= f(m_i, q_i) \\
q_{i+1} &= g(m_i, q_i)
\end{align*}
\]  

(1.1)

relating the mean magnetizations \(m_i\) and their mean squares \(q_i\) in layers or shells \(i\) of the tree far removed from the surface. In the limit \(i \to \infty\), \(m_i\) and \(q_i\) converge to a stable fixed point \(m, q\) of (1.1) which then become equivalent to the SK equations.

Since surface effects have been eliminated in the limit \(i \to \infty\), it is clear that \(m\) and \(q\) should be interpreted as local quantities rather than global or bulk expressions for the magnetization and its mean square, respectively. This is also the case for the nonrandom Ising model on a Cayley tree\(^8\), where it is known that the bulk properties\(^9\) are not described by local Bethe approximation expressions.

The free energy obtained from the SK equations should likewise be considered a local free energy, at least for the Cayley tree model. Negative local entropy is then mathematically possible but probably still physically undesirable. Nevertheless the random Ising model on a Cayley tree provides an instance where the SK equations are rigorously valid and also provides a possible interpretation of this theory as a local mean-field theory.

Our analysis certainly does not provide an exact solution to the original SK model nor does it provide a global mean-field theory of the spin glass state.