Transition in the Floquet Rates of a Driven Stochastic System

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Floquet theory is used to solve the Smoluchowski equation for a time-periodic system whose underlying dynamics exhibits a transition to deterministic chaos. For the stochastic version of this system, an abrupt transition occurs in the Floquet decay rates as parameters of the system are varied, leading to a much more rapid decay to the stationary state.

KEY WORDS: Fokker–Planck equation; Smoluchowski equation; Brownian motion in a potential well; Floquet theory; nonlinear response; driven Brownian particle.

1. INTRODUCTION

It is a great pleasure to contribute to this volume honoring Nico van Kampen, who has probably done more than anyone to bring clarity to the field of stochastic physics. In this paper, I consider a subject which has been a recurrent theme in his work, namely the response of a nonlinear system to a dynamic external field. This problem has become especially interesting because we now know that a nonlinear system coupled to a dynamic field will generally become chaotic as the parameters of the system are varied. A problem that has been little studied but is of growing interest is the behavior of a stochastic system whose underlying dynamics undergoes a transition to chaos. In this paper I consider such a system.

The problem I consider is that of a Brownian particle of mass $m$ and radius $R$ confined to an infinitely deep square-well potential with potential energy $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ otherwise. The square well is

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filled with a fluid with shear viscosity $\eta$ and the particle is driven by a monochromatic external force $f(t) = \varepsilon \sin(2\pi ft)$, where $\varepsilon$ is the amplitude of the force and $f$ is its frequency. The Langevin equation for the particle inside the well is

$$\frac{dv}{dt} = -\beta v + \varepsilon \sin(2\pi ft) - F(t)$$

(1.1)

where $\beta$ is the Stokes friction, $\beta = 6\pi R\eta/m$, and $F(t)$ is a delta-correlated white noise due to the many degrees of freedom of the fluid. [Note that $\langle F(t) F(t') \rangle = (k_B T/\beta) \delta(t - t')$, where $k_B$ is Boltzmann’s constant and $T$ is the temperature of the fluid. Hydrodynamic memory is neglected.] If no fluid is present, the classical mechanical version of this system$^{(1)}$ undergoes a transition to deterministic chaos in certain regions of the phase space as parameters of the external field are varied.

In this paper, I study the behavior of this stochastic system in the approximation where the friction is very strong so that the velocity relaxes to equilibrium on a time scale short compared to the period of the external field. The behavior of the system is then described by the Smoluchowski equation.$^{(2,3)}$ In Section 2, I write the Smoluchowski equation for the driven system. In Section 3, I use Floquet theory to determine the time evolution of the system, and in Section 4, I obtain the Floquet decay rates of the system.

2. DRIVEN PARTICLE IN AN INFINITE SQUARE WELL

Let us first consider the Smoluchowski equation for a particle confined to an infinitely deep square-well potential in the presence of white noise. The Smoluchowski equation for the particle in the interval $0 < x < L$ is

$$\frac{\partial P(x, t)}{\partial t} = D \frac{\partial^2 P(x, t)}{\partial x^2}$$

(2.1)

where $P(x, t)$ is the probability density of finding the particle at point $x$ at time $t$, the diffusion coefficient $D = k_B T/m\beta$, and the boundary conditions are

$$\frac{\partial P}{\partial x} \bigg|_{x = 0, L} = 0$$

to ensure that no probability flows through the walls. The solution to Eq. (2.1) takes the form

$$P(x, t) = \sum_{n = 0}^{\infty} c_n e^{-i\lambda_n t} \phi_n(x)$$

(2.2)