Derivation of the Boltzmann Equation for a Fermi Gas

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We consider the time evolution of a Fermi gas with two-body interaction. For an initial state \( \rho \) which is translation invariant and sufficiently clustering we put \( H = H_0 + \lambda V \), we take the limit \( \lambda \to 0, \tau \to \infty \) such that \( \lambda^2 \tau = \tau \) and show that
(a) the limiting state \( \rho_\tau \) does not depend on the \( p \)-point correlations of \( \rho \) for \( p > 2 \),
(b) \( \rho_\tau \) has vanishing \( p \)-point correlations for \( p > 2 \), and
(c) the two-point function that determines \( \rho_\tau \) satisfies the Boltzmann equation.

To avoid nonessential technical difficulties, we consider the case of a Fermi gas on a lattice.

KEY WORDS: Boltzmann equation; approach to equilibrium; \( \lambda^2 \tau \) limit.

1. INTRODUCTION

Hardly any equation in theoretical physics has evoked as much discussion and controversy as the Boltzmann equation. Much of the discussion was and still is centered around the fundamental question at what point and through what assumption the irreversibility in time was introduced. In the original derivation by Boltzmann an essential feature is the Stoszzahlansatz, an assumption about the lack of correlation between the velocities of two colliding particles. This assumption must be made not only at the initial time, but at all times. This clearly very undesirable aspect has triggered many attempts to find more satisfactory derivations.\(^2\)

A very similar situation exists with respect to the Pauli master equation. In the standard derivation of that equation, an assumption on the

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2 Of the many papers devoted to the derivation of the Boltzmann equation, we mention in particular the work of two groups, the group of Bogoliubov and coworkers and that of Prigonine and coworkers. See Ref. 1.
randomness of phases is made, again not just at the initial time but at all
times. In 1955 Van Hove\(^{(3)}\) published a paper in which a derivation of the
master equation was presented without repeated use of a random phase
assumption, but with a certain smoothness condition on the initial state. An
essential step in this approach is the recognition that in the case of many
degrees of freedom diagonal terms in the perturbation expansion are
predominant. Another important tool is the so-called \(\lambda^2 t\) limit, where the
coupling parameter \(\lambda\) tends to zero and the time \(t\) tends to infinity, in such
a way that \(\lambda^2 t\) has a finite limit \(\tau\).

One of the difficulties with Van Hove's derivation lies in the fact that
some of his basic assumptions are only valid for an infinite system, whereas
the formalism only applies to systems of a finite number of degrees of
freedom. Even the probabilities appearing in the master equation are not
well defined if the system is infinite.

In this paper a derivation is presented of the quantum Boltzmann
equation very much along the lines of Van Hove's paper. However, in our
case, essential use is made of a formalism, the so-called algebraic approach,
that is well suited for the treatment of infinite systems. Furthermore, the
distribution functions occurring in the Boltzmann equation are well defined
for such systems.

The system considered in this paper is a gas of Fermi particles with
two-particle interaction. In order to avoid technical difficulties that have to
do with convergence of integrals in momentum space, we deal with a
system where the particles are located on an infinite lattice instead of in a
continuous space. However, most of the arguments remain valid for the
continuous case, as long as one is willing to believe that those integrals
converge. The only assumptions made regard the initial state. They are: (1)
the initial state is homogeneous, and (2) the initial state satisfies a certain
cluster property. The first condition is one of mathematical convenience. It
seems feasible though more complicated to treat the nonhomogeneous case
along the same lines. The cluster property, i.e., an assumption on the decay
of correlation functions, is very essential. It is easy to construct examples of
nonclustering initial states that do not behave asymptotically according to a
Boltzmann equation.

With these initial conditions we prove that the one-particle distribution
function satisfies the Boltzmann equation in the \(\lambda^2 t\) limit. The physical
interpretation is clear. It means that a very weak interaction acts during a
long stretch of time. The variable \(\tau\) which is the limit of \(\lambda^2 t\) is to be
interpreted as a rescaled time parameter. It is the time variable occurring in
the Boltzmann equation. This method is the mathematical realization of the
old idea that, although noninteracting particles do not reach thermal
equilibrium, the introduction of an interaction, however slight, will bring
the system finally to thermal equilibrium.