Kinematics of the Forced and Overdamped Sine-Gordon Soliton Gas

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Motion of a driven and heavily damped sine-Gordon chain with a low density of kinks and tight coupling between particles is controlled by the nucleation and subsequent annihilation of pairs of kinks and antikinks. We show that in the steady state there are no spatial correlations between kinks or between kinks and antikinks. For a given number of kinks and antikinks all geometrical distributions are equally alike, as in equilibrium. A master equation for the probability distribution for the number of kinks on a finite chain is solved, and substantiates the physical reasoning in previous work. The probability distribution characterizing the spread along the direction of particle motion of a finite chain in equilibrium as well as in the driven overdamped case is derived by simple combinatorial considerations. The spatial spread of a driven chain in the thermodynamic limit does not approach a steady state; a given particle followed in time deviates as $t^{1/2}$ from its average forced motion. This result follows from the hydrodynamic equations for the dilute kink gas. Comparison is made with other recent results.

KEY WORDS: Sine-Gordon soliton gas; nucleation; annihilation; master equation; hydrodynamic equations; fluctuations; correlations.

1. INTRODUCTION

In this paper we study the statistical properties of the kink–antikink gas of the sine-Gordon equation, with emphasis on the forced and overdamped case, though some of our results are more general. One possible realization consists of a ring of torsion-coupled pendulums in a gravitational potential $V(1 - \cos \theta)$, and also under the action of an external torque $F$. In the

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overdamped case the time evolution of the displacement $\theta(x, t)$ of the pendulums is given by $^{1,2}$

$$\gamma \frac{\partial \theta}{\partial t} = -V \sin \theta + F + \kappa \frac{\partial^2 \theta}{\partial x^2} + \xi$$

(1)

where $\gamma$ is the damping constant and $\kappa$ is proportional to the coupling between adjacent pendulums. The system is connected to a thermal reservoir giving rise to fluctuations with a strength

$$\langle \xi(x, t) \tilde{\xi}(x', t') \rangle = 2\gamma kT \delta(t - t') \delta(x - x')$$

(2)

Equations (1) and (2) describe other physical systems $^{3,4}$ such as the Josephson junction transmission line $^{5,6}$ with negligible junction capacitance, or a chain of oscillators with phase coupling between adjacent oscillators and synchronized externally by a signal differing from the natural oscillator frequency. $^{7}$ Recent conference proceedings $^{3,4}$ demonstrate many other physical applications of the sine-Gordon equation, though for many of these the inertial term may not be neglected as we have done in Eq. (1).

For $|F| < V$ the potential $V(1 - \cos \theta) - F\theta$ possesses local minima at $\theta_s + 2\pi n$. At low temperatures the statistical mechanical properties in this field range are determined by two elementary types of excitations $^{8}$: small-amplitude relaxation modes ("overdamped phonons") around the stationary uniform states $\theta_\ast$, and large-amplitude excitations (kinks and antikinks) describing the transition from one Peierls valley at $\theta_\ast$ to an adjacent one at $\theta_\ast \pm 2\pi$. We adopt the notions "Peierls valley" and "Peierls hill" from the dislocation literature $^{8}$, the field in which the statistical mechanics of solitons was treated first.

We will call a transition from one Peierls valley to an adjacent one a kink if the first spatial derivative is positive, and an antikink if the first spatial derivative is negative. As a linguistic simplification, we will, on occasion, use the expression "kink" as a generic term, including both types of transitions. Under the action of a steady deterministic field, a kink travels with constant velocity $^{2} - u(F)$ in the presence of damping (Fig. 1), and an antikink travels with velocity $u(F)$ to the right. In the absence of a force the kink is at rest. (In contrast the kinks of the undamped sine-Gordon equation can travel with any fixed velocity smaller than the velocity of sound. The velocity for $F \to 0$, $\gamma \to 0$ depends on the order in which the limits $F \to 0$, $\gamma \to 0$ are taken.)

In this paper we study the statistical properties of a dilute kink gas. In the overdamped case and in the low-temperature regime, the kinematics of this gas is governed by two processes. If a kink and an antikink collide, they will annihilate (recombine). There is an attractive force between them; and since excess kinetic energy is immediately removed by the damping,