CONVECTIVE STABILITY OF A WEAKLY CONDUCTING LIQUID IN AN ELECTRIC FIELD

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In an inhomogeneously heated weakly conductive liquid (electrical conductivity \( \sigma \sim 10^{-12} \, \Omega^{-1} \, \text{cm}^{-1} \)) located in a constant electric field a volume charge is induced because of thermal inhomogeneity of electrical conductivity and dielectric permittivity. The ponderomotive forces which develop set the liquid into intense motion [1-6]. However, under certain conditions equilibrium proves possible, and in that case the question of its stability may be considered.

A theoretical analysis of liquid equilibrium stability in a planar horizontal condenser was performed in [2, 4]. Critical problem parameters were found for the case where Archimedean forces are absent [2]. Charge perturbation relaxation was considered instantaneous. It was shown that instability is of an oscillatory character. In [4] only heating from above was considered. Basic results were obtained in the limiting case of vanishingly small thermal diffusivity in the liquid (infinitely high Prandtl numbers). In the present study a more general formulation will be used to examine convective stability of equilibrium of a vertical liquid layer heated from above or below and located in an electric field. For the case of a layer with free thermally insulated boundaries, an exact solution is obtained. Values of critical Rayleigh number and neutral oscillation frequency for heating from above and below are found. Neutral curves are constructed. It is demonstrated that with heating from below instability of both the oscillatory and monotonic types is possible, while with heating from above the instability has an oscillatory character. Values are found for the dimensionless field parameter at which the form of instability changes for heating from below and at which instability becomes possible for heating from above.

§1. We will consider an incompressible inhomogeneously heated liquid located in an external constant homogeneous electric field. We will assume that temperature inhomogeneity is produced by external heating. We neglect Joulean heating and the induced magnetic field in view of the low electrical conductivity of the liquid. We will also assume that all liquid parameters except density, conductivity, and dielectric permittivity are temperature independent. The conductivity law is assumed to be Ohmic.

With the given assumptions, the system of equations describing convection in an electric field has the form

\[
\gamma \left[ \frac{\partial v}{\partial t} + (v \nabla) v \right] = -\nabla p + \eta \Delta v + \rho E - \frac{1}{\gamma_0} \nabla T - T + \gamma_0 \beta_0 T, \quad \text{div} \, v = 0
\]

\[
\frac{\partial T}{\partial t} + v \nabla T = \chi \Delta T
\]

\[
\frac{\partial \rho}{\partial t} + \text{div} (\rho E + \rho v) = 0, \quad \text{div} (\rho E) = 4 \gamma_0
\]

\[
\begin{align*}
\text{curl} \, E &= 0, \quad \sigma = \sigma_0 (1 + \beta_0 T), \quad \epsilon = \epsilon_0 (1 + \beta_0 T) \\
\beta_0 &= \frac{1}{\epsilon_0} \left( \frac{\partial \sigma}{\partial T} \right), \quad \beta_0 = -\frac{1}{\epsilon_0} \left( \frac{\partial \epsilon}{\partial T} \right)
\end{align*}
\]

Here \( \gamma_0 \) is liquid density and \( \rho \) is charge density; the remaining notation are those generally employed. The pressure \( p \) is reckoned from the mean value \( p_\alpha \), which includes hydrostatic and striction components;

\[ \nabla p_a = \frac{1}{\delta x} \nabla \left( \gamma \frac{\partial E}{\partial t} \right) + \gamma_{\text{eff}} \]  

(1.2)

We will assume further [7] that \( \beta c T \ll \beta_{T} T \ll 1 \). This allows us to consider further only terms linear in \( \beta_{T} T \), which produce the major contribution, and to neglect changes in dielectric permittivity with temperature.

Setting \( v = 0 \) and performing the operation curl on the first equation of (1.1), we obtain the general equilibrium condition: \( \textbf{E}_{0} \parallel \nabla T_{0} \parallel \textbf{g} \). We find equilibrium values for the temperature field and charge from Eq. (1.1):

\[
E_{0z} = -E_{00}(1 + \beta_{T} A_{0} z), \quad T_{z} = -A_{0} z, \quad \rho_{0} = -\frac{e}{4\pi} E_{00} \beta_{T} A_{0}
\]

where \( E_{00} \) is the field intensity in a vacuum, and the \( z \) axis is directed vertically upward. The equilibrium pressure value is determined by

\[
\frac{\partial p_{0}}{\partial z} = \gamma_{0} T_{0} g - \rho_{0} E_{0z}
\]

(1.4)

We will consider a small perturbation of the equilibrium state and denote deviations in temperature, pressure, charge density, and field from equilibrium values by \( T, p, \rho, E \), respectively, while the velocity of the motion developing will be denoted by \( v \). We linearize system (1.1) for small perturbations, choosing the following units for measurement of parameters: distance, the characteristic dimension of the cavity \( L \); electric field, \( E_{00} \); time, \( L^{2}/v \); velocity, \( \chi/L \); pressure \( \gamma_{0} \nu \chi/I^{2} \); temperature, \( A_{0} L \); charge \((e/4\pi)E_{00}\beta_{T}A_{0}\). We then obtain the dimensionless perturbation equations

\[
\frac{\partial v}{\partial t} = -\nabla p + \Delta v + RTn + RBp n \\
p = \frac{\partial T}{\partial t} - v = \Delta T, \quad \text{div} v = 0
\]

(1.5)

\[
P = \rho = \frac{\partial \rho}{\partial t}, \quad d e = S \rho, \quad \text{curl} e = 0
\]

\[
\dot{S} = 3 \frac{\beta_{T} A_{0} L}{\gamma_{0}}
\]

\[
P = \frac{\nu}{\chi}, \quad p_{T} = \frac{\nu v}{4\pi \sigma L^{2}}, \quad B = \frac{\nu E_{00} \beta_{T} A_{0}}{4\pi \gamma_{0} \sigma L}
\]

where \( n \) is a unit vector directed along the \( z \) axis.

Four dimensionless parameters appear in system (1.5): the Rayleigh number \( R \), the Prandtl number \( \text{Pr} \), the field parameter \( B \), and the relaxation parameter \( \text{Pr} \). The field parameter characterizes the ratio of Coulomb to Archimedean forces, while the relaxation parameter describes the ratio of relaxation time to characteristic development time of hydrodynamic disturbances and is an electrical analog of the Prandtl number. In the limiting case of instantaneous charge relaxation (\( \text{Pr} = 0 \)) system (1.5) transforms into the equations used in [2].

It should be noted that retention of only terms linear in \( \beta_{T} T \) in the final analysis means neglect of the effect of the electric field of the induced charge in comparison to the effect of the external field (analogous to the inductionless approximation in magnetic hydrodynamics).

§2. Attempting to find an exact solution of the problem, we will consider stability of equilibrium of an infinite planar vertical column of liquid with free thermal insulating boundaries. We will limit our examination to two-dimensional perturbations lying in the plane \( xz \). (The \( x \) axis is directed perpendicular to the boundaries.) It is known that in the absence of a field such perturbations possess the least stability [8]. Since the boundaries are planar (undeformed), thermally insulating, and are free surfaces, upon them, the following conditions must be fulfilled:

\[
v_{x} = 0, \quad \frac{\partial T}{\partial x} = 0 \quad (x = \pm L)
\]

(2.1)

\[
n_{s}(\sigma_{ik} - \sigma_{ik}^{(m)}) + n_{s}(T_{ik} - T_{ik}^{(m)}) = 0
\]

\[
\sigma_{ik} = -\rho \delta_{ik} + n \left( \frac{\partial v_{x}}{\partial x} + \frac{\partial v_{x}}{\partial x} \right)
\]

(2.2)

\[
T_{ik} = \frac{e}{4\pi} \left( E_{ik}E_{k} - \frac{1}{2} \delta_{ik}E^{2} \right) \quad (x = \pm L)
\]