Study of a Filtration Expanded to Include an Honest Time

M.T. Barlow
Department of Pure Mathematics, University College of Swansea, Singleton Park, Swansea, SA2 8PP UK

Summary. Let \((\Omega, \mathcal{F}, P)\) be a measurable space, and \(\{\mathcal{F}_t\}\) be a filtration on \((\Omega, \mathcal{F})\). Then, given a fixed honest time \(L\) a new filtration \(\{\mathcal{G}_t\}\) is defined, the smallest containing \(\{\mathcal{F}_t\}\) and for which \(L\) is a stopping time, and the martingales, semimartingales and stopping times of this new filtration are characterised.

0. Introduction

This paper presents a martingale approach to work on the decomposition of a process into its ‘past’ and ‘future’ relative to an honest random time. (See Millar [11], for a survey of the Markovian theory of such decompositions.)

Let \((\Omega, \mathcal{F}, P)\) be a complete probability triple, and \(\{\mathcal{F}_t, t \geq 0\}\) be a filtration consisting of sub-\(\sigma\)-fields of \(\mathcal{F}\) satisfying the ‘usual conditions’: that is, the filtration is right-continuous and increasing, and \(\mathcal{F}_0\) contains every \(P\)-null set in \(\mathcal{F}\). A random time on \((\Omega, \mathcal{F})\) is any \(\mathcal{F}\)-measurable map \(L: \Omega \rightarrow [0, \infty]\); a random time \(L\) is honest if for \(s \leq t\)

\[\{L \leq s\} = F_{st} \cap \{L \leq t\} \quad \text{for some } F_{st} \in \mathcal{F}_t.\]

This definition is equivalent (for a right-continuous filtration) to that given by Meyer, Smythe and Walsh in [10]. Most of the random times studied in connection with splitting-time theorems are honest: in particular optional, cooptional, and randomised coterminal times are all honest (see Millar [11]).

Let \(L\) be a fixed honest time, and for \(t \in \mathbb{R}^+\) define

\[\mathcal{G}_t = \{A \in \mathcal{F} : A = (E \cap \{L \leq t\}) \cup (F \cap \{L > t\}) \quad \text{for some } E, F \in \mathcal{F}_t\}.\]

Then \(\mathcal{F}_t \subseteq \mathcal{G}_t\), \(L\) is a \(\{\mathcal{G}_t\}\)-stopping time, and \(\{\mathcal{G}_t\}\) satisfies the usual conditions. We shall study the properties of the filtration \(\{\mathcal{G}_t\}\), and in particular of its martingales.

Let \(A_t = I\{t \geq L\}\), and let \(A^o\) and \(\hat{A}\) denote the optional and dual optional projections of \(A\) relative to \(\{\mathcal{F}_t\}\). In Section 2 we shall establish a few basic results concerning these processes. Section 3 is devoted to a study of \(\{\mathcal{F}_t\}\) and \(\{\mathcal{G}_t\}\) martingales: here is the main results of the section.
Theorem A. Let $M$ be a square integrable $\mathcal{F}_t$-martingale, and $M'$ be defined by

$$M'_t = M_t + \int_0^t \left[ (1 - A_{s-})(1 - A_{s-}^o)^{-1} - A_{s-} (A_{s-}^o)^{-1} \right] d\langle M, A^o - \dot{A} \rangle_s.$$ 

Then $M'$ is a square integrable $\mathcal{G}_t$-martingale.

As a corollary we show that every $\mathcal{F}_t$-semimartingale is a $\mathcal{G}_t$-semimartingale, providing a complement to a recent theorem of Stricker [12].

In Section 4 we investigate the 'measurable' structure of $\mathcal{G}_t$-progressive processes.

Theorem B. Let $T$ be a $\mathcal{G}_t$-stopping time. Then there exists a sequence $(S_n)$ of disjoint $\mathcal{F}_t$-stopping times such that

$$[T] \subset [L] \cup \bigcup_{i=1}^\infty [S_n].$$

In Section 5 and 6 we prove a martingale representation theorem for $\mathcal{G}_t$-martingales.

Theorem C. Suppose that $\{M^i : i \in I\}$ is a finite collection of continuous $\mathcal{F}_t$-local martingales, such that if $Y$ is any continuous $\mathcal{F}_t$-local martingale then there exist $\mathcal{F}_t$-previsible processes $C^i, i \in I$ such that

$$Y_t = \sum_{i \in I} \int_0^t C^i_s dM^i_s.$$ 

Then, if $Z$ is any continuous $\mathcal{G}_t$-local martingale, there exist $\mathcal{G}_t$-previsible processes $D^i, i \in I$, such that

$$Z_t = \sum_{i \in I} \int_0^t D^i_s d(M^i)_s.$$ 

To represent the jumps of $\mathcal{G}_t$-martingales we must use Jacod's theory of stochastic integrals relative to random measures.

Acknowledgements. I wish to thank my supervisor, Professor D. Williams, for suggesting this problem, and for various improvements to the style of this paper. Lemma 3.1 short-circuits a rather involved argument, leading to essentially the same results.

Note. Some of the results of Section 2 appear in Azéma [1]. T. Jeulin and M. Yor, in [8] and [13], written at the same time as this paper, have obtained most of the results of Sections 3 and 4, and go further in certain respects. The representation results in Sections 5 and 6 have not appeared before.

1. Notation and Preliminaries

It is not possible to give here more than a very brief account of the general theory of processes and martingales on which this paper is based: see the books.