Study of a Filtration Expanded to Include an Honest Time

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Summary. Let $(\Omega, \mathcal{F}, P)$ be a measurable space, and $\{\mathcal{F}_t\}$ be a filtration on $(\Omega, \mathcal{F})$. Then, given a fixed honest time $L$ a new filtration $\{\mathcal{G}_t\}$ is defined, the smallest containing $\{\mathcal{F}_t\}$ and for which $L$ is a stopping time, and the martingales, semimartingales and stopping times of this new filtration are characterised.

0. Introduction

This paper presents a martingale approach to work on the decomposition of a process into its ‘past’ and ‘future’ relative to an honest random time. (See Millar [11], for a survey of the Markovian theory of such decompositions.)

Let $(\Omega, \mathcal{F}, P)$ be a complete probability triple, and $\{\mathcal{F}_t, t \geq 0\}$ be a filtration consisting of sub-$\sigma$-fields of $\mathcal{F}$ satisfying the ‘usual conditions’: that is, the filtration is right-continuous and increasing, and $\mathcal{F}_0$ contains every $P$-null set in $\mathcal{F}$. A random time on $(\Omega, \mathcal{F})$ is any $\mathcal{F}$-measurable map $L: \Omega \to [0, \infty]$; a random time $L$ is honest if for $s \leq t$

$$\{L \leq s\} = F_{st} \cap \{L \leq t\} \quad \text{for some } F_{st} \in \mathcal{F}_t.$$

This definition is equivalent (for a right-continuous filtration) to that given by Meyer, Smythe and Walsh in [10]. Most of the random times studied in connection with splitting-time theorems are honest: in particular optional, cooptional, and randomised coterminal times are all honest (see Millar [11]).

Let $L$ be a fixed honest time, and for $t \in \mathbb{R}^+$ define

$$\mathcal{G}_t = \{ A \in \mathcal{F} : A = (E \cap \{L \leq t\}) \cup (F \cap \{L > t\}) \quad \text{for some } E, F \in \mathcal{F}_t \}.$$

Then $\mathcal{F}_t \subseteq \mathcal{G}_t$, $L$ is a $\{\mathcal{G}_t\}$-stopping time, and $\{\mathcal{G}_t\}$ satisfies the usual conditions. We shall study the properties of the filtration $\{\mathcal{G}_t\}$, and in particular of its martingales.

Let $A_t = I\{t \geq L\}$, and let $\hat{A}$ and $\hat{A}$ denote the optional and dual optional projections of $A$ relative to $\{\mathcal{F}_t\}$. In Section 2 we shall establish a few basic results concerning these processes. Section 3 is devoted to a study of $\{\mathcal{F}_t\}$ and $\{\mathcal{G}_t\}$ martingales: here is the main results of the section.
Theorem A. Let \( M \) be a square integrable \( \mathcal{F}_t \)-martingale, and \( M' \) be defined by

\[
M'_t = M_t + \int_0^t [(1 - A_{s-})(1 - A_{s-}^o)^{-1} - A_{s-}(A_{s-}^o)^{-1}] d\langle M, A^o - \hat{A}\rangle.
\]

Then \( M' \) is a square integrable \( \mathcal{G}_t \)-martingale.

As a corollary we show that every \( \mathcal{F}_t \)-semimartingale is a \( \mathcal{G}_t \)-semimartingale, providing a complement to a recent theorem of Stricker [12].

In Section 4 we investigate the 'measurable' structure of \( \mathcal{G}_t \)-progressive processes.

Theorem B. Let \( T \) be a \( \mathcal{G}_t \)-stopping time. Then there exists a sequence \( \{S_n\} \) of disjoint \( \mathcal{F}_t \)-stopping times such that

\[
[T] \subseteq [L] \cup \bigcup_{i=1}^\infty [S_n].
\]

In Section 5 and 6 we prove a martingale representation theorem for \( \mathcal{G}_t \)-martingales.

Theorem C. Suppose that \( \{M^i : i \in I\} \) is a finite collection of continuous \( \mathcal{F}_t \)-local martingales, such that if \( Y \) is any continuous \( \mathcal{F}_t \)-local martingale then there exist \( \mathcal{F}_t \)-previsible processes \( C^i, i \in I \) such that

\[
Y_t = \sum_{i \in I} \int_0^t C^i_s dM^i_s.
\]

Then, if \( Z \) is any continuous \( \mathcal{G}_t \)-local martingale, there exist \( \mathcal{G}_t \)-previsible processes \( D^i, i \in I \), such that

\[
Z_t = \sum_{i \in I} \int_0^t D^i_s d(M^i)_s.
\]

To represent the jumps of \( \mathcal{G}_t \)-martingales we must use Jacod's theory of stochastic integrals relative to random measures.

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Note. Some of the results of Section 2 appear in Azéma [1]. T. Jeulin and M. Yor, in [8] and [13], written at the same time as this paper, have obtained most of the results of Sections 3 and 4, and go further in certain respects. The representation results in Sections 5 and 6 have not appeared before.

1. Notation and Preliminaries

It is not possible to give here more than a very brief account of the general theory of processes and martingales on which this paper is based: see the books.