FLUID DYNAMICS

SLOW STEADY FLOWS OF A CONDUCTING FLUID AT LARGE HARTMANN NUMBERS

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We consider slow steady flows of a conducting fluid at large values of the Hartmann number and small values of the magnetic Reynolds number in an inhomogeneous magnetic field. The general solution is obtained in explicit form for the basic portion (core) of the flow, where the inertia and viscous forces may be neglected. The boundary conditions which this solution must satisfy at the outer edges of the boundary layer which develop at the walls are considered. Possible types of discontinuity surfaces and other singularities in the flow core are examined. An exact solution is obtained for the problem of conducting fluid flow in a tube of arbitrary section in a strong uniform magnetic field.

1. General solution. Let us consider slow motions of an electrically conducting incompressible fluid in a strong magnetic field. We will assume that the inertia and viscous forces can be neglected everywhere with the exception of narrow boundary layers; then the momentum equation for the flow core takes the form of the magnetostatic equation

\[ \nabla p = -\frac{1}{c} j \times H \quad \left( j = \frac{c}{4\pi} \text{rot} H \right). \] (1.1)

Let us examine steady or quasi-steady flows such that we can assume

\[ E = \nabla \phi. \] (1.2)

If the solution of (1.1) is known, the potential \( \phi \) and the velocity component \( v_\perp \) perpendicular to the magnetic field can be found from Ohm's law, whose projections in the direction of the magnetic field and in the plane perpendicular thereto have the form

\[ j_\perp = \sigma \frac{\partial \phi}{\partial s}, \quad j_\perp = \sigma \left[ (\nabla \phi)_\perp + \frac{v_\perp}{c} \times H \right]. \] (1.3)

Here \( \sigma \) is the electrical conductivity, and \( \partial / \partial s \) denotes a derivative taken in the direction of the magnetic field. Integrating the first equation of (1.3), we obtain an expression for \( \phi \) which contains as a term an arbitrary function of the coordinates which does not change along the magnetic lines of force. After finding \( v_\perp \) from the second equation of (1.3), we find the longitudinal component \( v_\parallel \) from the continuity equation

\[ \text{div} \mathbf{v} = 0. \] (1.4)

The expression thus found for \( v_\parallel \) contains the term \( fH \), where \( f \) is an arbitrary function which takes constant values on the magnetic lines of force. Thus, to each solution of the magnetostatic equations (1.1) there correspond explicit expressions for \( \phi \) and \( \mathbf{v} \).

In the following let us consider the case \( p \ll H^2 \).

Then (1.1) implies that the magnetic field in the first equation of (1.1) can be assumed to be irrotational and defined by external sources:

\[ \mathbf{H} = \nabla \alpha \quad (\alpha \text{ is a given function}). \] (1.5)

In this case the second of the equations in (1.1) is replaced by the equation

\[ \nabla \cdot j = 0. \] (1.6)

Then the pressure can be specified as an arbitrary function which does not vary along the magnetic lines of force; from (1.1) we find

\[ j_\perp = \frac{cH}{H^2} \times \nabla p. \] (1.7)

Substituting (1.7) into (1.6), after integration we find

\[ j_\parallel = H \int \frac{c}{H^2} (\mathbf{H} \times \nabla H^{-2}) \cdot \nabla p \, da + A \mathbf{H}. \] (1.8)

Here the integration is performed along the magnetic lines of force with respect to the quantity \( \alpha \), defined by (1.6); \( A \) is an arbitrary function, constant on each of the magnetic lines of force.

From (1.3) we obtain

\[ \sigma \phi = c \int \nabla H^{-2} (\mathbf{H} \times \nabla H^{-2}) \times \nabla p \, da + A \alpha + B. \] (1.9)

Here the integration is also performed along the magnetic lines of force, and \( B \) is an arbitrary function which does not vary along the magnetic lines of force. The conductivity \( \sigma \) was assumed constant for simplicity.

From the second equation of (1.3) we find

\[ v_\perp = -\frac{c^2}{\sigma H^2} \nabla p + \frac{cH}{H^2} \times \nabla \phi. \] (1.10)

Using (1.1), from the continuity equation (1.4) we find

\[ v_\parallel = H \int \frac{c}{H^2} (\nabla H^{-2} \times \mathbf{H}) \cdot \nabla \phi - \frac{c^2}{\sigma H^2} \nabla p \cdot \nabla H^{-2} - \frac{c^2}{\sigma H^2} \Delta p \, da + C \mathbf{H}. \] (1.11)

The integration is performed along the magnetic lines of force, \( \Delta \) is the Laplacian, and \( C \) is an
arbitrary function which is constant on each of the magnetic lines of force.

For given $H$, Eqs. (1.7)-(1.11) yield the general solution of (1.1), (1.3), (1.4), (1.6). This solution contains four arbitrary functions $p$, $A$, $B$, and $C$ which take constant values on each of the magnetic lines of force. Solutions of the type (1.7)-(1.11) in the case of a homogeneous magnetic field were previously obtained for certain particular forms of flows in [1-5].

2. Boundary conditions for the flow core. The boundary conditions for the flow core are the conditions at the outer edge of the boundary layer, which arises at the wall and transforms the boundary conditions at the wall to the boundary conditions for the flow core. In the wall boundary layer, in addition to the magnetic force and the pressure gradient, we must take viscosity into account, while the inertia forces can be neglected as before. These are known as Hartmann-type boundary layers and for certain forms of flows have been considered previously in [1,5,6]. Here we examine the boundary layer for an arbitrary mutual orientation of the wall, magnetic field, pressure gradient, and electric potential gradient in the flow core, although this complication does not lead to any new qualitative conclusions.

The equations describing the flow in the boundary layer have the form

$$
\frac{\partial \mathbf{v}}{\partial n} = \text{grad} p + \frac{\sigma}{c^2} \mathbf{H} \times \text{grad} \phi + \frac{\sigma H^2}{c^2} \mathbf{v} - \frac{\sigma}{c^2} \mathbf{H} (\mathbf{H} \cdot \mathbf{v}),
$$

$$
\frac{\partial \phi}{\partial n} = -\frac{1}{c} \left( \mathbf{H} \times \frac{\partial \mathbf{v}}{\partial n} \right)_n = 0. \tag{2.1}
$$

Here $n$ is the distance measured along the normal to the wall surface.

The first equation is obtained by substituting Ohm's law into the momentum equation, and the second is the continuity equation for the electric current.

Let us assume that the magnetic field component $H_n$ normal to the surface is nonzero. Let us also assume that in the boundary layer the velocity component normal to the surface is zero, while the tangential components of the potential gradient and the pressure gradient coincide with their values in the flow core.

We denote, respectively, by $u$, $\Phi$, and $P$ the differences

$$
u = v - v_0, \quad \Phi = \phi - \phi_0, \quad P = p - p_0,
$$

where $v_0$, $\phi_0$ and $p_0$ are the values of the velocity, potential, and pressure in the core.

Integrating the second equation of (2.1), assuming that $H$ does not change through the boundary layer thickness, we find

$$
\frac{\partial \Phi}{\partial n} = -\frac{1}{c} (\mathbf{H} \times \mathbf{u})_n. \tag{2.2}
$$

Using this expression and (2.1), we obtain the equation for $u$

$$
\mu \frac{\partial u}{\partial n^2} = \text{grad} p + \frac{\sigma}{c^2} H_n^2 u - \frac{\sigma}{c^2} H_n^2 (\mathbf{H} \cdot \mathbf{u})_n. \tag{2.3}
$$

Here $n$ is the vector normal to the surface.

Since $u_n = 0$, projecting (2.3) onto the normal to the surface, we find

$$
\frac{\partial P}{\partial n} = \frac{\sigma}{c^2} H_n (\mathbf{H} \cdot \mathbf{u}). \tag{2.4}
$$

Projecting (2.3) onto the plane tangent to the wall, we obtain the equation

$$
\frac{\partial u}{\partial n} = \frac{\sigma}{c^2} H_n^2 u. \tag{2.5}
$$

The solution of this equation, satisfying the boundary conditions at the wall and at infinity, has the form

$$
u = -v \exp \left( -\sqrt{\frac{\sigma H_n}{\mu \frac{\partial p}{\partial n}}} \right). \tag{2.6}
$$

Here $v$ is defined by (1.10) and by the condition $v_n = 0$

$$

\frac{\partial p}{\partial n} = \frac{\sigma}{c^2} \text{grad} \phi + \frac{\sigma H_n}{H_n^2} \frac{1}{\sigma \frac{\partial \phi}{\partial n}} - \frac{1}{c} (\mathbf{H} \cdot \text{grad} \phi)_n. \tag{2.7}
$$

We see from (2.6) that the ratio of the boundary-layer thickness to the characteristic flow dimension $L$ is of order $1/M$, where $M = \sqrt{\sigma HL/c^2 \mu}$ is the Hartmann number.

For $M \gg 1$, far from the walls, a flow core is formed, where (1.7)-(1.11) are valid. In the following let us assume that the condition $M \gg 1$ is always satisfied.

It is not difficult with the aid of Ohm's law and (2.4) and (2.2) to calculate the total current $I$ flowing in the boundary layer, nor to calculate the pressure variation $\delta p$ and the potential variation $\delta \phi$ across the boundary layer

$$
I = \frac{\sigma}{c} \int (\mathbf{u} \cdot \mathbf{H}) \cdot \mathrm{d}n = \sqrt{\frac{\mu}{\sigma} (\mathbf{v} \cdot \mathbf{n})}, \tag{2.8}
$$

$$
\delta p = \frac{\sigma}{c^2} H_n^2 \frac{1}{\sigma \frac{\partial \phi}{\partial n}} \int (\mathbf{H} \cdot \mathbf{u}) \cdot \mathrm{d}n = \sqrt{\frac{\mu}{\sigma} (\mathbf{H} \cdot \mathbf{v})}, \tag{2.9}
$$

$$
\delta \phi = \frac{1}{c} \int (\mathbf{H} \cdot \mathbf{u})_n \mathrm{d}n = \frac{1}{\sigma H_n^2} (\mathbf{H} \cdot \mathbf{v})_n, \tag{2.10}
$$

where the subscript $\tau$ denotes vector components tangent to the surface.

Equations (2.7)-(2.10) show that the quantities $I$, $\delta p$, and $\delta \phi$ have the following orders of magnitude:

$$
I \sim \frac{c \delta p}{H_n} \sim \frac{c \delta \phi}{\mu \mathbf{E}_\perp} \sim \frac{1}{M} \max \left\{ \frac{\sigma}{H_n} \mathbf{g}L \right\}. \tag{2.11}
$$

Here $H_n$, $\mathbf{E}_\perp$ are the characteristic values of the magnetic field intensity and the transverse component of the electric field. We note that variation in pressure and potential in the boundary layers cannot lead to marked variation of velocity in the flow core. In fact, by varying the boundary values of the potential and pressure we can alter the gradients of these quantities in the flow core by the magnitudes $\delta \phi/L$ and $\delta \phi/\mu \mathbf{E}_\perp$, and it is not difficult to see that the corresponding