APPLICATION OF THE CHAPMAN-ENSKOG METHOD TO THE CASE OF A BINARY TWO-TEMPERATURE GAS MIXTURE

V. S. Galkin


An interesting property of the flows of a binary mixture of neutral gases for which the molecular mass ratio \( \epsilon = m/M \ll 1 \) is that within the limits of the applicability of continuum mechanics the components of the mixture may have different temperatures. The process of establishing the Maxwellian equilibrium state in such a mixture divides into several stages, which are characterized by relaxation times \( \tau \) which differ in order of magnitude. First of all, the component of the light component reaches equilibrium, then the heavy component, after which equilibrium between the components is established \([1]\). In the simplest case the relaxation times differ from one another by a factor of \( \sqrt[3]{\epsilon} \).

In the case of a cold ionized plasma the Chapman-Enskog method is quite easily extended to the case of the two-temperature mixture \([3]\), since the Landau collision integral is used, which decomposes directly with respect to \( \epsilon \). In the Boltzmann cross collision integral, the quantity \( \epsilon \) appears in the formulas relating the velocities before and after collision, which hinders the decomposition of this integral with respect to \( \epsilon \), which is necessary for calculating the relaxation terms in the equations for temperatures differing from zero in the Euler approximation \([4]\) (the transport coefficients are calculated considerably more simply, since for their determination it is sufficient to account for only the first (Lorentzian \([5]\)) terms of the decomposition of the cross collision integrals with respect to \( \epsilon \)). This leads to the use in \([4]\) for obtaining the equations of the considered continuum mixture of a specially "constructed" model kinetic equation (of the Bhatnagar-Krook type) which has an undetermined degree of accuracy.

In the following we use the Boltzmann equations to obtain the equations of motion of a two-temperature binary gas mixture in an approximation analogous to that of Navier-Stokes (for convenience we shall term this approximation the Navier-Stokes approximation) to determine the transport coefficients and the relaxation terms of the equations for the temperatures. The equations in the Burnett approximation, and so on, may be obtained similarly, although this derivation is not useful in practice.

With exception of the temperatures \( T_1 \) and \( T_2 \) are, respectively, the light and heavy gas temperatures, expressed in energetic units), we shall denote the quantities relating to the heavy and light gases by the corresponding upper and lower case symbols: \( F \), \( f \) are the distribution functions; \( R = MN \), \( \rho = mn \) are the mass densities; \( P = NT^2 \), \( p = nT^1 \) are the scalar pressures; \( P_{ij} \), \( P_{ij} \) are stresses; \( Q_i \), \( q_i \) are thermal fluxes; \( R = MN \), \( p = mn \) are the corresponding upper and lower case symbols: \( F \), \( f \) respectively, the light and heavy gas temperatures, the average velocity of the heavy component

\[
V_1 = \frac{1}{N} \int F_{1e} dC. \tag{1}
\]

Here \( \xi \) is the absolute velocity of the heavy molecule, so that

\[
\int F_{1e} dC = 0, \quad \nu = \int \frac{f_1 e}{e} dC \neq 0, \tag{2}
\]

\[
W_i = \left( \frac{M}{2T_2} \right) c_i, \quad w_i = \left( \frac{m}{2T_1} \right) c_i, \quad N F^* = \left( \frac{2T_2}{M} \right) F_i, \quad n_i^* = \left( \frac{2T_1}{m} \right)^{1/3} f_i. \tag{3}
\]

We note that usually \([5]\) the molecular peculiar velocities are calculated relative to the mass average velocity, which in the present case differs from \( V \) by the amount \( \Delta \ll \sqrt{\epsilon} \). Therefore, in the zeroth approximation these velocities are equal.

We write the Boltzmann equations in the form

\[
\frac{dF}{dt} = \Xi = \int F_{1e} dC, \quad \frac{dF}{dt} = \gamma = J_{FF} + I, \tag{4}
\]

\[\Xi = J_{FF}, \quad \gamma = J_{FF} - I, \quad I = N \int (f - f) e db db de. \]

Here we have the used notations from \([5]\), \( I \) is the zero with respect to \( \epsilon \) (Lorentzian \([5]\)) value of the cross collision integral \( J_{FF} \), the analogous integral in the equation for \( F \) is equal to zero on the basis of the conservation law for the number of particles. In the right sides of the equations for \( F \) and \( f \) there appear quantities of order \( 1/\tau_1 \gg 1 \). If \( \epsilon \ll 1 \), so that \( \epsilon/\tau_1 F \sim 1 \), then (as shown below) the terms of the left sides are of order unity. It is precisely in this "limiting" sense that the quantities \( \Xi \) and \( \sigma \) are used in the following. Under these conditions, in the first (in the Chapman-Enskog sense) approximation with account for the arbitrary nature of \( N \) the solutions of Eqs. (4) will be Maxwellian functions with different temperatures,

\[
F_{1e} = \frac{\pi^{-\sqrt{\epsilon}e^{-w^2}}}{\pi^{-\sqrt{\epsilon}e^{-w^2}}}, \quad j_{0e} = \frac{\pi^{-\sqrt{\epsilon}e^{-w^2}}}{\pi^{-\sqrt{\epsilon}e^{-w^2}}}. \tag{5}
\]

For future use, by the usual method we obtain from (4) with account for (1), (2) the system of equations of motion of the gas mixture

\[
\frac{DN}{Dt} + N \frac{\partial V_1}{\partial x_1} = 0, \quad \frac{Dn}{Dt} + n \frac{\partial V_1}{\partial x_1} + \frac{\partial n v_i}{\partial x_i} = 0, \tag{6}
\]

\[
R \frac{DV_1}{Dt} + \frac{\partial P}{\partial x_1} + \frac{\partial P_{ij}}{\partial x_j} = \Phi_1, \quad \Phi_1 = M \int C_i \xi dC, \tag{7}
\]

\[
\frac{3}{2} N \frac{DT_1}{Dt} + (P_{0i} + P_{ij}) \frac{\partial V_1}{\partial x_j} = 0. \tag{8}
\]
\[
\frac{\partial Q_i}{\partial x_i} = E, \quad E = \frac{M}{n} \sum C \nu d \xi,
\]
\[
\frac{3}{2} n \frac{D T_i}{D t} + (p \nu_{ij} + \nu_{ij} \frac{\partial V_j}{\partial x_i}) + \frac{\partial q_i}{\partial x_i} - \frac{3}{2} T_1 \frac{\partial n_i}{\partial x_i} = -E.
\]

In the left side of the last equation we have dropped the term \( p \nu_{ij} D V_i / D t \), which is negligibly small in the approximation used in the following (in [4] the last term of the left side of this equation is also incorrectly dropped). System (6) differs from the analogous system for the single-temperature binary mixture in the structure of the equations; in place of the energy equation we have here two equations for the temperatures with the relaxation terms \( E \).

In the first (Euler) approximation \( \Phi_1(f_0, F_0) = 0 \), the integral
\[
E = \frac{M}{n} \int f_0 \phi_0(C^2 - C^0) \sum b d \nu d \xi d \phi d \chi d \Phi = 16 \pi \eta (T_1 - T_2) \frac{n N_n}{(1 + \varepsilon)^2} \times
\]
\[
\left\{ e^{-\eta \eta((x + \varepsilon) + \delta \Phi_0^0(\eta))} \right\} \eta d \eta. \tag{7}
\]

Here \( \Phi_0^0(\eta) \) is a function of the argument in the brackets, the functions \( \Phi_1^0(1(\eta)) \) and the quantities \( \Omega_1^0(1) \), which are used later, have the same sense as in [5]. If the repulsive force between the molecules is equal to \( x_{12} \varepsilon^\nu \), then
\[
\Phi_0^0 = A_1(\varepsilon) \left[ \frac{\eta_{12}}{m} \left( 1 + \varepsilon \right)^{\nu^0} \right],
\]
\[
\chi_1 = \frac{2}{\nu - 1}, \quad \chi_2 = \frac{\nu - 5}{(2 - \nu)(1 - 1)}, \quad T = \frac{T_2}{T_1}. \tag{8}
\]

We represent \( E \) in the form
\[
E = E_0(1 + \varepsilon E_1 + \varepsilon E_2). \tag{9}
\]

Here \( E_2 \) is determined by the second (Navier-Stokes) approximation; \( E_0 \) and \( E_1 \) are found by expanding the integral (7) in terms of \( \varepsilon \),
\[
E_0 = 16 \pi \eta (T_1 - T_2) \frac{n N_n}{(1 + \varepsilon)^2} \times
\]
\[
\left\{ e^{-\eta \eta((x + \varepsilon) + \delta \Phi_0^0(\eta))} \right\} \eta d \eta, \tag{10}
\]

Hence, in the case (8) we obtain
\[
E_0 = 8 \pi \eta (T_1 - T_2) \frac{n N_n}{(1 + \varepsilon)^2} \times
\]
\[
\left\{ e^{-\eta \eta((x + \varepsilon) + \delta \Phi_0^0(\eta))} \right\} \eta d \eta, \tag{11}
\]

From (10) follows the condition for which the mixture components may have different temperatures:
\[
E_0 / NT_2 \sim 1 \quad \text{or} \quad \varepsilon \epsilon(n \Omega_0^0(1)) \sim 1. \tag{12}
\]

This last condition is easily expressed in terms of the gas parameters that are customary for aerodynamics (see also [4], the characteristic flow time \( t \sim 1 \)). With account for (10), (11), we obtain from (6) the equations of the first approximation
\[
\frac{D \ln n}{D t} = -\frac{\partial V_i}{\partial x_i}, \quad \frac{D \ln N}{D t} = -\frac{\partial V_i}{\partial x_i}, \quad \frac{R D V_i}{D t} = -\frac{\partial P}{\partial x_i}, \quad \frac{3}{2} \frac{D \ln T_2}{D t} = -\frac{\partial V_i}{\partial x_i} + \frac{E_0}{NT_2}, \quad \frac{3}{2} \frac{D \ln T_1}{D t} = -\frac{\partial V_i}{\partial x_i} + \frac{E_0}{n T_1}. \tag{13}
\]

Substituting the functions (5) into the left sides of the Boltzmann equations (4) and eliminating the total derivatives \( D(\ldots)/D t \) with the aid of (12), we obtain the following expression:
\[
F_0 \left\{ \left( w^2 - \frac{5}{2} \right) c_1 \frac{\partial \ln T_3}{\partial x_i} + 2[W] \frac{\partial V_i}{\partial x_i} + \right.
\]
\[
+ \frac{E_0}{n T_2} \left( \frac{2}{3} w^2 - 1 \right) \right\} - \Sigma_0 \tag{14}
\]
\[
F_0 \left\{ \left( w^2 - \frac{5}{2} \right) \frac{\partial \ln T_1}{\partial x_i} + 2[W] \frac{\partial V_i}{\partial x_i} + \right.
\]
\[
+ \frac{E_0}{n T_1} \left( 1 - \frac{2}{3} w^2 \right) \right\} - \Sigma_0 \tag{15}
\]

The problems of determining the transport properties of the light and heavy gases separate. The stresses and thermal fluxes for the heavy gas are the same as for the same single-component gas. We denote the second approximation to \( f \) by \( f_0 \). From (14) it follows that
\[
q = -A \eta c_1 \frac{\partial \ln T_3}{n \partial x_i} - B \frac{[w]}{n} \frac{\partial V_i}{\partial x_i} - D \frac{c_1 \partial \ln T_3}{n \partial x_i} + K. \tag{16}
\]

The coefficients \( A, B, D, K \) depend on \( \omega, T_1, N/n \). The equations for the first three coefficients are solved by the same technique as for the single-component gas, and the second terms of the right sides of the equations are integrated just as in the case of a Lorentzian gas [5]. For example, the equation for \( A \) takes the form
\[
f_0 \left( \omega^2 - \frac{5}{2} \right) = \int f_0 \phi_0^{\delta} \left( \phi_0^{\delta} \right) g \sum b d \nu d \phi d \chi d \Phi d \eta d \xi d \phi d \chi d \Phi d \eta d \xi d \phi d \chi d \Phi d \eta d \xi d \phi d \chi d \Phi d \eta d \xi d \phi d \chi d \Phi d \eta d \xi d \phi d \chi d \Phi d \eta d \xi d \phi d \chi d \Phi d \eta d \xi d \phi d \chi d \Phi. \tag{17}
\]

Here and in the following \( \{ H \} = H_1 + H_2 - H_1' - H_2' \).