CONSIDERATION OF LONGITUDINAL DIFFUSION IN FLOW WITHIN A CHANNEL

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The stationary problem of convective diffusion in a channel with absorbent walls is considered. It is assumed that a Poiseuille flow exists. Two methods are employed in the solution, the method of separation of variables, and the method of expansion in eigenfunctions of the corresponding problem with piston profile (expansion method). It is established by comparison with independently obtained solutions for high Peclet number that for the first eigenfunctions and eigenvalues the expansion method gives satisfactory results over the entire Peclet-number range. For approximate calculation of subsequent eigenfunctions and eigenvalues a modification of the smooth asymptotic expansion method is used. The results are used to calculate matter flow density on the wall, to evaluate the length of the entrance region, and to obtain an analytical expression for the limiting Nusselt number in terms of the Peclet number.

1. Formulation of the Problem and Method of Separation of Variables. The mathematical formulation of the problem is the following:

\[
(1 - y^2) \frac{\partial C}{\partial x} = \varepsilon \left( \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} \right) \quad \left( \varepsilon = \frac{D}{V h} \right)
\]

\[
C|_{x=1} = \Phi(y), \quad C|_{x=0} = 0, \quad C|_{y=1} = 0, \quad (0 < x < \infty; \quad -1 \leq y \leq 1)
\]

Here \(C(x, y)\) is the concentration of the material; \(D\) the diffusion coefficient, \(h\) the channel half-width (the unit of length), \(V\) the velocity on the flow axis. The variables \(C, x,\) and \(y\) are dimensionless.

It is assumed that \(\Phi(y) = \Phi(-y)\). Then one can restrict oneself to the interval \(-1 \leq y \leq 0\) and take the boundary conditions of Eq. (1.1) in the following form:

\[
C|_{x=1} = 0, \quad \frac{\partial C}{\partial y} \bigg|_{y=-1} = 0
\]

In the separation-of-variables method, a solution of Eq. (1.1) is sought in the form [1, 2, 3]

\[
C(x, y) = \sum_{k=0}^{\infty} A_k(\varepsilon) Y_k(\varepsilon, y) \exp \left( -\varepsilon \lambda_k^2 x \right)
\]

\[
Y_k(\varepsilon, y) = \Phi \left( \frac{1 - \lambda_k^2 - \varepsilon \lambda_k^2 y}{2}, \frac{1}{2}, \lambda_k^2 y^2 \right) \exp \left( -\frac{1}{2} \lambda_k^2 y^2 \right)
\]


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where $\lambda_k(\varepsilon)$ are the roots of the equation

$$
\Phi \left( \frac{1 - \varepsilon^2 \lambda^2}{4} - \frac{1}{2} \lambda \right) = 0
$$

(1.6)

Here $\Phi(\sigma, c, z)$ is the degenerate hypergeometric function.

Comencing from Eq. (1.6), in [3] the asymptotic $\lambda_0(\varepsilon)$ was found for $\varepsilon \to 0$ and $\varepsilon \to \infty$, and in [2] the quantities $\lambda_k(\varepsilon)$ ($k=0, 1, 2$) were calculated. Equation (1.3) is not a Sturm–Liouville equation. The existence of eigenfunctions and eigenvalues can be proved [4, 5], but the question of completeness of the eigenfunction system remains open. Even if it is assumed that $\varphi(y)$ can be expanded in functions $Y_k(\varepsilon, y)$, the problem of finding $\tilde{X}_k(\varepsilon)$ arises, since the "orthogonality condition" for the functions $Y_k(\varepsilon, y)$

$$
\int_{-1}^{1} [\varepsilon^2(\lambda_n^2 + \lambda_{m}^2) + 1 - y^2] Y_m(\varepsilon, y) Y_n(\varepsilon, y) dy = 0 \quad (m \neq n)
$$

(1.7)

does not permit determination of the coefficients $\tilde{X}_k(\varepsilon)$. Moreover, due to the complexity of working with the degenerate hypergeometric function, it is desirable to have a satisfactory representation of the solution in terms of simpler functions.

2. Expansion of the Problem with Piston Profile in Eigenfunctions. In [6, 7] the solution of Eq. (1.1) was sought in the form

$$
C(x, y) = \sum_{k=0}^{\infty} C_k(x) \cos(\mu_k y)
$$

(2.1)

Based on the representation of Eq. (2.1) the method of solving Eq. (1.1) presented below will be termed the expansion method. No estimates of the accuracy or range of applicability of this method were given in [6, 7]. Below, such an estimate will be made by direct comparison with the method of separation of variables.

In the expansion method for $C_k(x)$ we obtain an infinite system of differential equations

$$
C_{k,\nu} - \alpha_{k} e^{-i \mu_k} C'_{k} - \mu_k^2 C_k = \sum_{\nu=0}^{\infty} \alpha_{k,\nu} C_{\nu}'
$$

(2.2)

$$
C_k(0) = a_k, \quad C_k|_{x \to \infty} = 0 \quad (k, \nu = 0, 1, 2, \ldots)
$$

The constants $a_k$ and $\alpha_{k,\nu}$ are found from the relationships

$$
Q_{\nu} = \int_{-1}^{1} q(y) \cos(\mu_k y) dy
$$

(2.3)

$$
\alpha_{k,\nu} = \int_{-1}^{1} (1 - y^2) \cos(\mu_k y) \cos(\mu_\nu y) dy = \begin{cases} 
\frac{8(-1)^{\nu+1} \mu_k}{(\mu_k - \mu_\nu)^2 (\mu_k + \mu_\nu)^2} & (k \neq \nu) \\
\frac{2}{3} + \frac{1}{2 \mu_k^2} & (k = \nu)
\end{cases}
$$

(2.4)

The method proposed in [6] for solution of Eq. (2.2) is based on the assumption that the right side of Eq. (2.2) may be regarded as a perturbation. Then in the zeroth approximation Eq. (2.2) is uncoupled, and the method of successive approximation is then used. The small parameter in which the expansion is conducted is a quantity of the type $\alpha_{k,\nu}/\alpha_{k,0}$. We write the solution of Eq. (1.1) corresponding to the first approximation

$$
C(x, y) \approx \sum_{k=0}^{\infty} \exp(\gamma_k x) F_k(y)
$$

(2.5)

$$
F_k(y) = a_k \cos(\mu_k y) - \cos(\mu_k y) \sum_{\nu=0}^{\infty} a_\nu A_{k,\nu} + a_k \sum_{\nu=0}^{\infty} A_{\nu,0} \cos(\mu_\nu y)
$$

(2.6)

$$
A_{k,\nu} = \alpha_{k,\nu} \left[ a_{k,\nu} - a_{k,0} e^{-i \gamma_k (\mu_k^2 - \mu_\nu^2)} \right]^{-1} \quad (k \neq \nu), \quad A_{\nu,0} = 0
$$

(2.7)

$$
\gamma_k = \frac{[a_{k,0} + i a_k - 4 \mu_k^2 \varepsilon]}{(2\varepsilon)^{-1}}
$$

(2.8)