The self-similar problem of the motion of a cold gas subjected to an instantaneous impulse is considered. A solution is constructed in the neighborhood of the known exact solution for a gas with a specific heat ratio $\gamma = 1.4$ [1-4]. An analytic expression is obtained in this neighborhood for the dependence of the index of self-similarity $n$ on the parameter $h$, which is related to the adiabatic index $\gamma$ by $h = (\gamma + 1)/((\gamma - 1)$. The results of a numerical calculation of $n$ versus $h$ are compared with the analytic expression.

1. The problem of the motion of a gas subjected to a short-lived shock was first formulated in [5] in the following manner. Consider a half-space bounded by a vacuum and filled with an ideal gas; at first the gas density is constant, and the pressure and velocity equal zero. At the initial time an instantaneous impulse is applied to the gas from the direction of the vacuum, and a shock wave propagates through the gas, which expands into the vacuum. Since there are no characteristic length and time scales in the problem, it is clear that the motion is self-similar.

The Lagrangian equation for the one-dimensional motion of a gas is, in standard notation,

$$\frac{\partial v}{\partial t} + \frac{\partial p}{\partial m} = 0, \quad \frac{\partial \rho}{\partial t} + \rho v \frac{\partial p}{\partial m} = 0, \quad \frac{d}{dt}\left(\frac{p}{\rho^2}\right) = 0 \quad (1.1)$$

System (1.1) must satisfy the boundary conditions at the leading edge of a strong shock (since in front of the leading edge $p = 0$) and the conditions for the expansion of the gas into the vacuum (the conditions at the free surface).

Following [5, 6], the self-similar solution of (1.1) is sought as

$$p = \Delta \rho_0 M^{-\eta} \eta(\eta), \quad v = \sqrt{\Delta M^{-\eta}} \eta(\eta), \quad \rho = \rho_0 \eta(\eta) \quad (1.2)$$

where $\eta = m/M$ is a self-similar independent variable, $n$ is the index of self-similarity, $\Delta$ is a constant determined by the instantaneous impulse, and $M = M(t)$ is the Lagrangian coordinate of the leading edge of the shock ($0 \leq \eta \leq 1$, $1 < n < 2$).

After substituting the variables [1, 6]

$$f = \eta x^n, \quad w = \eta^{1-n} x^n u \quad (1.3)$$

system (1.1) reduces to an ordinary differential equation and the quadratures

$$\frac{du}{dx} = \frac{h(h+1)u^n + (2+n) \sqrt{h(h-1)} u x - 2(h+1) h(h-1) u - 2(h-1) n x}{2h^n u + 2(2+n) \sqrt{h(h-1)} x - 2h(h-1)(2(h+1) - (h-1) n)} \quad (1.4)$$

The index of self-similarity $n$ is determined from the condition that the integral curve of (1.4) pass through the point $B(x=1, \, u=\sqrt{(h-1)/h})$ which corresponds to the shock, and the singular point of this equation $C(x=h+1, \, u=-2\sqrt{(h-1)/h})$ which is a saddle point for all values of $h$.

2. The substitution

$$u(x) = \sqrt{(h-1)/h} \, (x, \, y), \quad x=1+hx_1, \quad y=1-y_1,$$  \hspace{1cm} (2.1)

transforms the points $B$ and $C$ into the points $B_1(0,0)$ and $C_1(1,3)$. Thus, in the plane $x_1y_1$ the coordinates of the points under consideration are independent of $n$ and $h$, and the subsequent investigation is simplified.

In the variables of (2.1) Eq. (1.4) becomes

$$\frac{dy_1}{dx_1} = \frac{1}{2} \left( \frac{-n(h+1)y_1^2 + (n-2h)y_1 - (2h-3n-2nh)y_1 + h(2-n)}{h(2+n)x_1^n - nhx_1 + [2+n+2h(n-1)]x_1 - ny_1 + 2(n-1)} \right),$$  \hspace{1cm} (2.2)

and the known [1-4] exact solution of (1.4) for $h=6, \, n=\frac{4}{3}$ is written more simply: $y_1 = 3x_1$.

When $h$ varies in the neighborhood $h=6$, the index of self-similarity $n$ varies in some neighborhood $n=\frac{4}{3}$. In this case the exact solution is deformed somehow, remaining unchanged at $B_1$ and $C_1$ since their coordinates are independent of $n$ and $h$. Thus, we seek the solution of (2.2) in the neighborhood of the known exact solution, setting

$$y_1 = 3x_1 + eg(x_1), \quad h=6+ae, \quad n=\frac{4}{3}+\delta e$$  \hspace{1cm} (2.3)

where $\delta$ is a small dimensionless parameter, and $a$ and $b$ are constants which must be determined from the boundary conditions. It is clear that the unknown functions $g(x_1)$ must satisfy the following homogeneous conditions:

$$g(0)=0, \quad g(1)=0$$  \hspace{1cm} (2.4)

which ensure that the integral curve of (2.2) passes through $B_1$ and $C_1$.

Substituting (2.3) into (2.2) and omitting terms higher than first order, we obtain the linear differential equation

$$g'' + \frac{1}{2} \frac{3x_1+4}{(x_1-1) (x_1+4/3)} g + \frac{3(2a+27b)x_1^2 - (8a+27b)x_1 + 2a - 34b}{24(x_1-1) (x_1+4/3)} = 0$$  \hspace{1cm} (2.5)

whose solution is

$$g = \frac{1}{(x_1-1)^3} \left[ \frac{2a+27b}{20} z^4 + \frac{8a+81b}{12} z^3 - \frac{7(5a+27b)}{12} z^2 - \frac{49(16a-27b)}{2.6} z - \frac{7^2(2a-27b)}{4.6^4} \right]$$  \hspace{1cm} (2.6)

where $z=x_1^{4/3}$, and $C$ is a constant of integration.

The solution (2.6) is governed by the boundary conditions (2.4). The second condition in (2.4) is satisfied in the following manner: for $x_1=1$ the numerator and its third derivative in (2.6) are set to zero; the constant of integration is determined from the first condition

$$C = 14\sqrt{7}(4a+9b)/15\sqrt{6}$$  \hspace{1cm} (2.7)

and the second condition is satisfied similarly. Satisfying the first condition in (2.4) and taking (2.7) into account, we obtain the ratio of the unknown constants

$$\tau = \frac{a}{b} = \frac{2}{9} \frac{28\sqrt{7}-67}{101-14\sqrt{7}}$$  \hspace{1cm} (2.8)

From (2.3) setting $\delta=(h-6)/a$ we obtain the analytic equation for determining the index of self-similarity $n_*$, and also the solution of (2.2) in some neighborhood $h=6$. 

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