ASYMPTOTIC THEORY FOR CALCULATING HEAT FLUXES NEAR THE CORNER OF A BODY

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Izv. AN SSSR, Mekhanika Zhidkosti i Gaza, Vol. 4, No. 5, pp. 53–60, 1969

A method is presented for calculating the distribution of the thermal fluxes, friction stresses, and pressure near the corner point of a body contour in whose vicinity the outer supersonic flow passes through an expansion wave. The method is based on a study of the asymptotic solutions of the Navier-Stokes equations as the Reynolds number \( R \) approaches infinity for the flow region in which the longitudinal gradients of the flow functions are large, invalidating conventional boundary layer theory. This problem was examined in part in [1], in which the distribution of the friction and pressure in a region with length on the order of a few thicknesses of the approaching boundary layer was obtained in the first approximation. The leading term of the expansion for the thermal flux to the surface of the body vanishes for a value of the Prandtl number equal to unity and for other values of the Prandtl number does not match directly with its value in the undisturbed boundary layer.

The thermal-flux distribution is obtained for values of the Prandtl number \( \sigma \) approaching unity. For this purpose it was necessary to consider a more general double passage to the limit as \( \sigma \rightarrow 1 \) and \( \epsilon \rightarrow 0 \) for a finite value of the parameter \( R = [\sigma (\sigma - 1)]^2 \left[ - \ln \epsilon^{1/3} / \epsilon \right]^{1/2} \) characterizing the ratio of the effects of thermal conduction, viscous dissipation, and convection. The solution obtained previously [1] corresponds to the particular case \( |B| \rightarrow \infty \) and therefore for actual values of \( R = 10^3 \rightarrow 10^5 \), \( \sigma \rightarrow 0, 7 \) overestimates considerably the effect of the dissipative term on heat transfer, although even in first approximation it describes the pressure distribution well and the friction distribution satisfactorily. For smooth matching of the solutions with the corresponding flow functions in the undisturbed boundary layer it was necessary to introduce a flow region with “free interaction” for the expansion flow. Equations and boundary conditions which describe the flow as a whole are presented. Examples are given of numerical calculations and comparison with experiment.

1. Figure 1 shows a schematic of the flow in question. Far from the body surface in region 1 there is uniform supersonic flow with pressure \( p_0^* \). Let the body length from the leading edge or the stagnation point be \( l^* \). Then the thickness of the undisturbed boundary layer is equal in order of magnitude to \( \epsilon l^* \), where \( \epsilon = \epsilon \left[ R^{-1/3} \right] = \rho \ddot{p}_0^* / \mu \left( \rho_0^*, \mu_0^* \right) \) and \( \ddot{p}_0^* \) are, respectively, the dimensional density, dynamic viscosity, and velocity in the undisturbed supersonic flow. As the supersonic flow passes through the expansion wave near the corner, the pressure in the flow changes over the length \( \epsilon l^* \) by an order of magnitude. Therefore in region 2 with the thickness \( \epsilon l^* \) we must introduce the following dimensionless variables:

\[
S = s^* / l^*, \quad N = n^* / l^*, \quad \Psi = \psi^* / \epsilon \dot{p}_0^* u_0^* l^*;
\]

where the degree symbol denotes dimensional quantities. The asymptotic expansions for the flow functions have the form

\[
\begin{align*}
\rho^*(s^*, n^*; \epsilon) &= \rho_0^* \rho_3(S, N) + \ldots, \\
\rho^*(s^*, n^*; \epsilon) &= \rho_0^* u_0^* \rho_3(S, N) + \ldots, \\
u^*(s^*, n^*; \epsilon) &= u_0^2 u_0^* (S, N) + \ldots, \\
v^*(s^*, n^*; \epsilon) &= u_0^3 v_3 (S, N) + \ldots.
\end{align*}
\] (1.1)

The subscripts indicate the region number.

If we substitute (1.1) into the Navier-Stokes equations, we obtain the Euler equations for the first terms of the expansion in terms of \( \epsilon \). Joining of the solutions in regions 1 and 2 yields the connection between the magnitude of the pressure and the inclination of the velocity vector in region 2 as \( N \rightarrow \infty \) in the form of the usual relation for an expansion wave in a supersonic flow. The other boundary condition is \( v_3(S, 0) = 0 \). To satisfy the no-slip conditions \( u^*(s^*, 0; \epsilon) = 0 \) it is necessary to examine the flow in region 3, in which the leading viscous and inertial terms of the Navier-Stokes equations have the same order of magnitude. It follows from this condition that the thickness of the region must be of order \( \epsilon^{3/2} / \rho_0^* l^* \). Since on the basis of the condition for the joining of regions 2 and 3 the velocity in the new region remains of order \( u_0^* \), the value of the stream function \( \psi^* \) must be of order \( \epsilon^{3/2} / \rho_0^* l^* \). The dimensionless independent variables and the asymptotic forms of the other flow functions have the form

\[
\begin{align*}
\dot{S} &= s^* / l^*, \quad \Psi = \psi^* / \epsilon \dot{p}_0^* u_0^* l^*, \\
u^*(s^*, n^*; \epsilon) &= \rho_0^* u_0^* \rho_3(S, \Psi) + \ldots, \\
v^*(s^*, n^*; \epsilon) &= \rho_0^* \rho_0^* \rho_3(S, \Psi) + \ldots, \\
\rho^*(s^*, n^*; \epsilon) &= \rho_0^* \rho_0^* \rho_3(S, \Psi) + \ldots, \\
v^*(s^*, n^*; \epsilon) &= \rho_0^* \rho_3(S, \Psi) + \ldots. \quad (1.2)
\end{align*}
\]

It was shown in [1] that for a value of the Prandtl number \( \sigma \) equal to unity the stagnation enthalpy remains constant across the region 3 in first approximation. Therefore, in this paper we study the first term of the expansion for \( H^*(s^*, n^*; \epsilon) \), which is variable in region 3, as the Prandtl number approaches unity:

\[
H^*(s^*, n^*; \epsilon) = u_0^3 [H(S, 0) + \alpha(\epsilon) H_3(S, 0) + \ldots].
\]

Here \( H(S, 0) \) is a given constant quantity corresponding to the surface temperature of the body; \( \alpha(\epsilon) \) is an unknown function which will subsequently be determined from joining the expansions obtained in region 3 with the expansions which will be obtained for the other flow regions. At the body surface the boundary conditions have the form

\[
w_3(S, 0) = v_3(S, 0) = H_3(S, 0) = 0.
\]

Upon substituting (1.3) into the Navier–Stokes equations and transformation to the Mises variables, we obtain the equations

\[
\rho_0^* \frac{\partial H_3}{\partial S} = \rho_0^* \frac{\partial}{\partial \Psi} \left( \frac{\rho_0^* \mu_0^*}{\sigma} \frac{\partial H_3}{\partial \Psi} \right). \quad (1.3)
\]
In the following we assume that

\[ \frac{\partial n_3}{\partial s} = \frac{v_3}{u_3}, \quad \frac{\partial n_2}{\partial s} = \frac{4}{\rho \mu_3}. \]  

(1.3) (cont'd)

In the following we assume that

\( (s - 1) / a(\epsilon) \to 0 \text{ as } \epsilon \to 0 \)

\( (s - 1, a(\epsilon) \to 0 \text{ as } \epsilon \to 0) \).

It was shown in [1] that the pressure disturbances in region 2 decay as \( S^{-2} \). When \( S \to -\infty \) the velocity disturbances in region 2 are equal in order of magnitude to the quantities in the layer near the body and far from the body.

In fact, the flow in region 2 satisfies the Bernoulli equation. The Bernoulli constant for each stream filament is determined by joining the solution in region 2 with the solution for the undisturbed boundary layer.

Near the body, where the velocity in the undisturbed boundary layer approaches zero, in the disturbed region 2 the velocity disturbances are on the order of magnitude of the quantities in the layer near the body and far from the body.

\[ \frac{\partial p_3}{\partial \Psi} = 0, \quad \rho_3 \mu_3 \frac{\partial u_3}{\partial s} + \frac{\partial p_3}{\partial s} = \rho_3 \mu_3 \frac{\partial u_3}{\partial \Psi}, \]

\[ \frac{\partial n_3}{\partial s} = \frac{v_3}{u_3}, \quad \frac{\partial n_2}{\partial \Psi} = \frac{4}{\rho \mu_3}. \]  

(1.4)

Equations for region 4 are obtained by substituting (1.4) into the Navier-Stokes equations. The results which are needed later have the form

\[ \frac{\partial n_2}{\partial \Psi} = 0, \quad \frac{\partial p_2}{\partial \Psi} = 0, \quad \rho_2 + \rho_2(\psi) u_2(\psi) u_2 = 0, \]

(1.5)

where \( \rho_2(\psi) \) and \( u_2(\psi) \) are the density and velocity profiles in the undisturbed boundary layer for \( s^* = 0 \). The first formula (1.5) shows that a change of the boundary layer thickness of order \( s^{5/4} \) does not take place in region 4, and the coordinate of the streamline can change by this magnitude only by means of a shift due to a change of the thickness of the lower layer 5. According to (1.5), in the leading term of the layer 4 there is no transverse pressure differential and the Bernoulli equation (in linearized form) holds.

According to the estimates presented above for the perturbations of the pressure, velocity, and stream function, in region 5 the independent variables and the asymptotic representations of the stream functions have the form

\[ s = s^* / \epsilon^{3/\ell}, \quad \Psi = \psi^* / \epsilon^{3/\ell} \psi_0(\psi); \quad \rho = \rho_0(s, 0) + \ldots, \]

\[ \rho = \rho_0(s, 0) + \ldots, \quad \mu = \mu_0(s, 0) + \ldots, \]

\[ H(s^*, \psi^*; \epsilon) = \rho_0(s, 0) [H_0(s, 0) + \epsilon \mu_0(s, 0) + \ldots], \]

\[ H(s^*, \psi^*; \epsilon) = \rho_0(s, 0) \mu_0(s, 0) \frac{\partial}{\partial \Psi} [\psi_0(\psi, \psi)]_0. \]

Upon substituting (1.6) into the Navier-Stokes equations we obtain the equations of the incompressible layer, in which the gas temperature and density equal their values in the undisturbed layer near the body surface

\[ \rho_0(s, 0) \frac{\partial u_5}{\partial s} + \frac{\partial p_5}{\partial s} = \rho_0(s, 0) u_5(\epsilon) \frac{\partial}{\partial \Psi} [\rho_0(s, 0) \mu_6(0) \frac{\partial u_5}{\partial \Psi}], \]

\[ \frac{\partial n_5}{\partial s} = \frac{v_5}{u_5}, \quad \frac{\partial H_5}{\partial s} = \frac{\rho_0(s, 0) \mu_5(0) \frac{\partial}{\partial \Psi} [\psi_0(\psi, \psi)]_0, \]

(1.7)

The boundary conditions at the body surface are

\[ u_4(s, 0) = u_4(s, 0) = H_4(s, 0) = 0. \]

(1.8)

From the joining of the solutions in regions 5 and 4 we obtain the outer boundary conditions for region 5

\[ u_5(s, \Psi) \to \left[ \frac{2(\Psi - \rho_0)}{\rho_0(s, 0)} \right]^{1/6}, \]

\[ H_5(s, \Psi) \to b \left[ \frac{2\Psi}{\rho_0(s, 0)} \right]^{1/6}, \]

\[ n_5(s, \Psi) = n_4(s) \text{ for } \Psi \to \infty. \]

(1.9)