N-SOLITON-TYPE SOLUTIONS OF THE SELF-DUAL YANG–MILLS EQUATIONS IN M^4

R. K. Bullough, B. S. Getmanov, and P. M. Sutcliffe

We have investigated by computer in the case N = 2 the dynamics of an N-soliton type (N-monopole-type) solution of the self-dual Yang–Mills equations in Minkowski space-time M^4 found previously. Even for N = 2 this solution involves choices of up to 18 parameters. For “head-on” collisions an exotic dynamics already develops, involving the disappearance of the monopoles, their exchange, and/or the appearance of additional features, spheres, and discs.

1. INTRODUCTION

The numerical investigation of soliton-type collisions in 1+1 dimensions [1, 2] provided invaluable insights which in the case of the Korteweg–de Vries equation led to its exact analytical solution [3]. Atiyah and Hitchin [4] point out correspondences and differences between the interaction of the soliton solutions of an integrable system such as Korteweg–de Vries in 1+1 dimensions and the interaction of BPS monopole solutions of the Bogomolny equations in R^3 [4]. In an earlier letter [5], one of us (BSG) reported an exact analytically expressed solution of the self-dual Yang–Mills (sdYM) equations in Minkowski space-time M^4 which was of N-soliton (N-monopole) type. This solution, formula (13) of [5], depended on 8N parameters parameterizing certain vectors n^a (see below) together with a set of discrete parameters e^a = ±1. It was impossible to infer the dynamical behaviour of this solution by inspection and a brief computer investigation of the simplest “head-on” collision for the simple case of N = 2 already revealed a picture very far from the usual picture of 2-dimensional elastic scattering. We have now been able to carry out more extensive numerical investigations, which reveal striking dynamical behaviours even for N = 2. We use this letter to make a preliminary report of these numerical results.

A dimensional reduction of the sdYM equations in 2+2 dimensions produces [6] a Yang–Mills–Higgs–Bogomolny equation in 2+1 dimensions. For an SU(1,1) gauge group a ’t Hooft-like ansatz has been used [6] to construct in 2+1 dimensions a one-monopole-like solution and an N-soliton-type (N-monopole-like) solution comparable with the 1- and N-soliton-type solutions derived in 3+1 dimensions, very much as in [5], in this letter. In [6], two of us (BSG and PMS) report a numerical investigation of dynamical behaviours corresponding to the cases N = 1 and N = 2. The numerical work for 3+1 dimensions reported in this letter is based on the experience already gained in 2+1 dimensions in [6], and the dynamical results for 3+1 dimensions which we report in this letter show a surprising measure of agreement with those reported already for 2+1 dimensions in the [6].

2. ANALYTICAL SOLUTIONS

The sdYM equations are three matrix-valued differential equations on the matrix-valued potential A^a, μ = 0, 1, 2, 3. Define the YM field by

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]; \] (1)

then the sdYM equations in Minkowski space-time M^4 are

\[ F_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}; \] (2)

in which \( \epsilon_{\mu\nu\alpha\beta} \) is the totally antisymmetric 4-tensor. These equations are complex number valued for the standard case of an SU(N) group but may be considered as real for the noncompact group SL(N,C) [7]. Here we restrict ourselves to the simplest case of the SU(2) group, so that A^a = -iA^a_\mu \sigma_i/2. The \( \sigma_i \) are the Pauli matrices.
In principle, there are at least two ways of obtaining exact $N$-soliton-type solutions of these equations: These are (1) by making a successful ansatz, and (2) by applying some version of an inverse scattering formalism. The latter would rest on the integrability of the sdYM equations as it is established in $\mathbb{R}^4$ in terms of a Lax pair in [8]. In this letter we adopt the first approach much as already sketched in [5]; we hope to use the second approach, making a direct generalization of the UNILOF scheme technique for the two-dimensional case [9, 10, 11], in a future paper.

Following [5] we shall construct here regular spherically symmetric one soliton-type solutions of Eqs. (2) (monopole solutions in $M^4$), and we shall give a corresponding $N$-soliton-type generalization. To study the dynamics of such solutions we need to write them in an arbitrary frame. So, following [5], we first introduce here in $M^4$ an orthonormal basis of three space-like vectors $k_i^\alpha$ ($i=1,2,3; \alpha=0,1,2,3$);

\[ k_i^\alpha k_j^\beta = -\delta_{ij} \delta^{\alpha\beta}, \]

as well as a time-like vector $n_i^\alpha$.

Following these definitions we immediately derive useful identities, such as $c_{ij} k_i^\alpha k_j^\beta = c_{ij} v^\alpha v^\beta$ and the completeness condition $k_i^\alpha k_i^\beta = n_0 n_\alpha - g_{\alpha\beta}$, where $g_{\alpha\beta}$ is the metric tensor. We next use these identities to construct the following important objects:

(I) An antisymmetric tensor $R^i_{\mu\nu} = -\epsilon^{ijk} k_j^\alpha k_k^\beta$, its dual $*R^i_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\alpha\beta} R^j_{\alpha\beta}$, the self-dual tensor $\eta^i_{\mu\nu} = R^i_{\mu\nu} + i \epsilon_{\mu\nu\alpha\beta} R^j_{\alpha\beta}$, and the anti-self-dual tensor $\bar{\eta}^i_{\mu\nu} = R^i_{\mu\nu} - i \epsilon_{\mu\nu\alpha\beta} R^j_{\alpha\beta}$. It is easy to check that the two $\eta$ tensors introduced this way satisfy in $M^4$ the identities established by 't Hooft in [12]; in fact, in the standard reference frame ($k_i^\mu = -\delta_i^\mu$, $n_\mu = (1,0,0,0)$) our tensors actually coincide with 't Hooft's tensors. So $\eta^i_{\mu\nu}$, $\bar{\eta}^i_{\mu\nu}$ are 't Hooft's tensors in an arbitrary reference frame, i.e., they are the covariant forms of 't Hooft's tensors.

(II) The scalar variable

\[ w = \sqrt{(k_\mu x_\mu)^2} = \sqrt{s^2 - x_\mu^2} = \sqrt{-\frac{1}{2} (n_\mu x_\nu - x_\nu n_\mu)^2} = \sqrt{-c_\mu^2}. \]  

We denote $s = n_\mu x_\mu$, $\xi_\mu = n_\mu s - x_\mu$, $x_\mu = x_\mu - x_\mu$, and the $x_\mu$ define the position of a localized object in $M^4$. In the standard frame (the rest frame), $w = \bar{F} = |F - \bar{F}_0| = |(x_i - x_{0i})^2|$. We need $w$ to construct the spherically symmetric functions for an arbitrary frame in covariant form. The derivative of $w$, $\partial_\mu w \equiv \frac{x_\mu}{w}$, is a unit vector: $w_\mu^2 = 1$.

In terms of the objects defined in (I) and (II), we can construct regular spherically symmetric solutions of the sdYM equations, Eqs. (2), and their $N$-soliton-type generalization in an arbitrary frame of $M^4$. It is convenient to start from