SOLITON DYNAMICS FOR NEAR INTEGRABLE
DIFFERENTIAL-DIFFERENCE EQUATIONS

Russell L. Herman

In this talk, we consider both the spectral and the direct perturbation methods for studying perturbations of an integrable
differential-difference equation, the Toda lattice. Both methods employ the formalism of inverse scattering to represent
the corrections in terms of an appropriate basis of squared eigenfunctions.

1. INTRODUCTION

The perturbed Toda lattice equations,

\[ a_{n_t} = a_n(b_{n+1} - b_n) + \epsilon F_n, \]

\[ b_{n_t} = 2(a_n^2 - a_{n-1}^2) + \epsilon G_n, \]

have been studied by many researchers over the last decade. The first discussion of a dissipative Toda lattice was given
in the work of Nagashimi and Amagishi [10], who were investigating wave propagation in a nonlinear transmission
line. They treated the perturbation of a small-amplitude soliton by transforming the perturbed Toda equations to a
damped Korteweg-deVries (KdV) equation. To obtain the effects of the perturbation on the soliton parameters, they
had employed the adiabatic results of Karpman and Maslov [5]. However, Karpman and Maslov had since published
their well-known non-adiabatic results, which account for the reaction of the shelf on the velocity of the soliton [5].
Therefore, the results of Nagashimi and Amagashi should be modified. In 1979, Yajima had set up a perturbation theory
for the Toda lattice, using inverse scattering theory [13]. He had studied the time dependence of the perturbed scattering
data along the same lines as was done in the work of Kaup and Newell for nonlinear evolution equations associated with
the Zakharov-Shabat spectral problem [6, 7]. In the same year Kako applied Yajima’s formalism to the study of wave
propagation in a set of perturbed Toda equations describing a nonlinear transmission line [4]. Kako’s results for the
amplitude behavior agreed with Nagashimi and Amagashi’s continuum approximation for small amplitudes. However,
the predicted velocity of the soliton was found to disagree with numerical results. A similar problem had occurred in
the study of the damped KdV equation [7, 2]. Kaup and Newell had used the spectral approach to study the behavior of
the damped KdV soliton [7]. They found that the damped soliton traveled faster than the undamped soliton. This was
also in disagreement with numerical simulations and the predictions of other researchers [5].

In this paper we will present a discussion of two treatments of the perturbed Toda equations, which can be used to
determine the first-order corrections to the perturbed Toda equations. These results, in turn, have been used to extend
the work of Kako.

2. ONE-SOLITON SOLUTION OF TODA EQUATIONS

It is well known that the Toda equations,

\[ a_{n_t} = a_n(b_{n+1} - b_n), \quad b_{n_t} = 2(a_n^2 - a_{n-1}^2), \]

are integrability conditions for the Lax pair [1, 11]

\[ a_n v_{n+1} + a_{n-1} v_{n-1} + b_n v_n = lv_n, \]

\[ v_{n_t} = a_n v_{n+1} - a_{n-1} v_{n-1}, \]

Mathematical Sciences Department, The University of North Carolina at Wilmington, Wilmington, NC 28403
USA. E-mail: herman@seq.cms.uncwil.edu. Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 99,
where \( \lambda = (z + z^{-1})/2 \). We are interested in the Jost solutions of Eq. (4), \( \phi_n(z), \psi_n(z), \tilde{\phi}_n(z) = \phi_n(z^{-1}), \) and \( \tilde{\psi}_n(z) = \psi_n(z^{-1}) \), satisfying the boundary conditions

\[
\phi_n(z) \to z^n, \quad n \to +\infty, \quad \psi_n(z) \to z^{-n}, \quad n \to -\infty.
\]

As (4) is a second-order difference equation, any two of these solutions can be written as a linear combination of the other two. Therefore, we have

\[
\psi_n(t; z) = \alpha(z)\phi_n(t; z) + \beta(z)\phi_n(t; z),
\]

\[
\tilde{\psi}_n(t; z) = \tilde{\alpha}(z)\phi_n(t; z) - \tilde{\beta}(z)\phi_n(t; z),
\]

where

\[
\tilde{\alpha}(z) = \alpha(z^{-1}), \quad \tilde{\beta}(z) = -\beta(z^{-1}), \quad \alpha(z)\tilde{\alpha}(z) + \beta(z)\tilde{\beta}(z) = 1.
\]

Bound state solutions, corresponding to the discrete spectrum \( \{z_k, k = 1, \ldots, N\} \), are obtained for \( \alpha(z_k) = 0 \). The Jost solutions are no longer independent in this case. Evaluating equations (8) at \( z = z_k \) yields

\[
\psi_n(z_k) = \beta_k\phi_n(z_k), \quad \phi_n(z_k) = \tilde{\beta}_k\psi_n(z_k).
\]

Furthermore, we can define the normalization constants

\[
C_k^{-1} = \sum_{n=-\infty}^{\infty} \phi_n^2(z_k), \quad \tilde{C}_k^{-1} = \sum_{n=-\infty}^{\infty} \psi_n^2(z_k).
\]

From the asymptotic behavior of the Jost functions, we can define two sets of scattering data:

\[
S_- = \left\{ \left( \frac{\beta(z)}{\alpha(z)}, (z_k, C_k), k = 1, \ldots, N \right) \right\},
\]

\[
S_+ = \left\{ \left( \frac{\tilde{\beta}(z)}{\alpha(z)}, (z_k, \tilde{C}_k), k = 1, \ldots, N \right) \right\}.
\]

The inverse scattering transform is a canonical transformation between solutions of the unperturbed Toda lattice equations and one of these sets of scattering data. This is useful for solving the Cauchy problem for the Toda lattice, which is discussed in [1, 11].

In this paper we are interested in the one-soliton solution of the Toda lattice equations and the corresponding Jost solutions. These solutions correspond to a single zero, \( z_1 \), of \( \alpha(z) \) and are reflectionless, \( \beta(z) = 0 \). In the discussion to follow we will take

\[
z_1 = e^{-\omega} \text{ real}, \quad |z_1| < 1.
\]

In this case we have

\[
\alpha(z) = \frac{z_1 - z}{1 - z z_1}, \quad \beta(z) = 0,
\]

giving us a zero reflection coefficient, as expected. The Jost solutions can be found in terms of hyperbolic functions as

\[
\phi_n(z) = \frac{K_n z^{n+1}}{1 - z z_1} \left[ \frac{1}{z} - z_1 + \sinh \omega (\tanh \theta_n - 1) \right],
\]

\[
\psi_n(z) = \frac{K_n z^{-n}}{1 - z z_1} \left[ z_1 - z - \sinh \omega (\tanh \theta_n - 1) \right],
\]

where \( K_n \) is given as

\[
K_n^{-2} = \frac{\text{sech} \theta_n}{z_1 \text{sech} \theta_{n-1}}
\]

and \( \theta_n = n\omega - \delta \). In terms of the soliton parameters, \( \omega \) and \( \delta \), we have

\[
\beta_1 = e^{\omega + 2\delta}, \quad \tilde{\beta}_1 = e^{-\omega - 2\delta}.
\]

Finally, a one-soliton solution of the Toda equations (3) can be derived from the inverse scattering theory. It is given by the equations

\[
4a_n^2 - 1 = \sinh^2 \omega \text{sech}^2 \theta_n, \quad b_n = -\frac{1}{2} \sinh^2 \omega \text{sech} \theta_n \text{sech}(\theta_n - \omega).
\]