SOLVABILITY OF THE DERIVATIVE NONLINEAR SCHRODINGER EQUATION AND THE MASSIVE THIRRING MODEL

Jyh-Hao Lee

Here we review some results of J.-H. Lee of the $N \times N$ Zakharov–Shabat system with a polynomial spectral parameter. We define a scattering transform following the set-up of Beals-Coifman [2]. In the $2 \times 2$ cases, we modify the Kaup–Newell and Kuznetsov–Mikhailov system to assure the normalization with respect to the spectral parameter. Then we are able to apply the technique of Zakharov–Shabat for the solitons of NLS to our cases. We obtain the long-time behavior of the equations which can be transformed into DNLS and MTM in laboratory coordinates respectively.

1. $N \times N$ ZAKHAROV–SHABAT SYSTEM WITH A POLYNOMIAL PARAMETER

Let $J = \text{diag}(id_1, id_2, \ldots, id_N)$, $d_1 < d_2 < \cdots < d_N$, $i = \sqrt{-1}$, $\Sigma = \{z : \text{Im}(z^n) = 0\}$. Let $q_i, i = 1, 2, \ldots, n$, be $M_N(\mathbb{C})$-valued functions and $q_i$ be in the Schwartz class for all $i$. We consider the following $N \times N$ system:

$$
dM/dz = z(J, M) + (z^n - q_1 + z^{n-2}q_2 + \cdots + q_n)M, \quad z \notin \Sigma,
$$

where $M(z, z)$ is bounded and absolutely continuous, $M(x, z) \to I$ as $x \to -\infty$. The potentials $q_i$ must satisfy the constraints described in [16, 25]. These constraints enable us to have $M(x, z)$ normalized at $z = \infty$. Thus, we may pose the inverse problem properly. Following the set-up of Beals–Coifman, we may solve $M$ via the Fredholm integral equation or we may apply the wedge product technique to solve the columns of $M$ via Volterra equations. We refer the readers to the papers Beals and Coifman [1-5] and J. H. Lee [15-20].

Let $\Omega_+ = \{z : \text{Im}(z^n) > 0\}$, $\Omega_- = \{z : \text{Im}(z^n) < 0\}$. For the generic potential $(q_1, q_2, \ldots, q_n)$, $M$ has the following properties: (i) For any $z \in \Sigma \setminus \{0\}$, there is a unique $v(z)$ such that for all $x$, $M^+(x, z) = M^-(x, z) \exp(xz^n) v(z) \exp(-xz^n J)$, where $M^+ = \text{limit of } M$ on $\Sigma$ from $\Omega_+$; (ii) $M(x, \cdot)$ has a finite number of poles at $D = \{z_1, z_2, \ldots, z_L\}$. For any $z_j$, there is a matrix $v(z_j)$ such that $\text{Res}(M(x, \cdot), z_j) = N(x, z_j) \exp(xz_j^n J) \cdot v(z_j) \exp(-xz_j^n J)$, where $N(x, z)$ is the regular part of $M(x, z)$ near $z = z_j$. (iii) The map $(q_1, q_2, \ldots, q_n) \to (v(z_1), v(z_2), \ldots, v(z_L))$ is injective. If the potential has the symmetric conditions $q_j^* = -q_j$, by uniqueness $M^*(x, \bar{z}) = M(x, z)^{-1}$. It follows that $v^*(\bar{z}) = v(z), \quad z \in \Sigma; \quad v^*(\bar{z}_j) = -v(z_j)$ for $j = 1, 2, \ldots, L$.

Given scattering data $v$ satisfying the constraints as described in [20, p. 328], the inverse problem amounts to solving an analytic factorization problem (the Riemann–Hilbert problem) with a parameter $x$. We may use the argument of Beals–Coifman [2] and the property of weighted-boundedness of a Hilbert transform to show that the inverse problem is solvable if $v(z) - I$ is small. Choosing a suitable function space, we have shown that the inverse problem is solvable for $v$ in an open and dense subset [17, 18]. The argument of Beals–Coifman establishes a Fredholm property [4]. We may use this to show that if the scattering data satisfies the symmetric conditions $v^*(\bar{z}) = v(z), \quad z \in \Sigma, \quad v^*(\bar{z}_j) = -v(z_j)$, then the inverse problem is soluble.

REMARKS. The analytic results of the scattering and the inverse scattering for the generalized Zakharov–Shabat system with rational spectral parameter have been worked out by X. Zhou. Zhou also gave a different formulation of $M$ in the inverse problem [35–37].

2. EVOLUTION EQUATIONS

Suppose that the evolution of the scattering data is given by

$$
\begin{align*}
   d\nu(z, t)/dt &= z^k [J, \nu(z, t)]; \\
   d\nu(z, t)/dt &= z_j^k [J, \nu(z_j, t)], \quad z_j \text{ is fixed for all } j = 1, 2, \ldots, L, \quad k \text{ is a multiple of } n.
\end{align*}
$$

Then the evolution equation for $q_j$ is as follows:

$$
dq_j/dt = -(\{[q_{k-1}, G_1] + [q_{k-2}, G_2] + \cdots + [q_1, G_n] + [J, G_{n+1}]\}, \quad k = 1, 2, \ldots, n.
$$

where \( G_j \) is the coefficient of \( 1/z^j \) in the asymptotic expansion of \( MJM^{-1} \) as \( z \to \infty \) [15-20].

Equation (2.2) is of the form \( U_t - V_x + [U, V] = 0 \), where \( U = z^n J + z^{n-1} q_1 + z^{n-2} q_2 + \cdots + q_n \), \( V = z^k J + z^{k-1} G_1 + \cdots + z G_{k-1} + G_k \).

REMARKS. (i) The wedge product technique of constructing \( M \) enables us to control the \( L^\infty \)-norm of \( M \) by the \( L^2 \)-norm of \((Q, P)\) in the case \( dM/dx = z^2 [J, M] + (zQ + P)M \). By this result, we have the global existence of the dissipative evolution equation associated with this system [20]. (ii) We note that the symmetric conditions of the scattering data described in Section 2 are invariant under the evolution (3.1). Hence we have the global existence of the associated evolution equation with \( q_j^* = -q_j \).

3. DNLS

Here we review some results of Lee [19]. Let \( Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix} \) with \( Q^* = -Q \),

\[
P = Q(adJ)^{-1}Q = \begin{pmatrix} qr/2i & 0 \\ 0 & -qr/2i \end{pmatrix};
\]

we consider the system \( dM/dx = z^2 [J, M] + (zQ + P)M \), Im \( z^2 \neq 0 \), where \( M(., z) \) is bounded and absolutely continuous, and \( M(x, z) \to I \) as \( x \to \infty \). By the constraint \( P = Q(adJ)^{-1}Q \), we have \( M(x, z) \to I \) as \( |z| \to \infty \). Hence the scattering data satisfies the following constraints:

\[
\begin{cases}
\frac{v^*(z)}{z^2} = v(z) \\
v^*(z_j) = -v(z_j) \\
v^*(z) = v(z) \\
v^*(z_j) = -v(z_j)
\end{cases}
\]

and \( v(z_j) \) is of the form \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) when \( \text{Im}(z_j^2) > 0 \); \( v(z_j) \) is of the form \( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \) when \( \text{Im}(z_j^2) < 0 \). Here \( \sigma \) is an automorphism on \( M_2(C) \) defined by \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sigma = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \). If \( v(z, t), v(z_j, t) \) evolve as

\[
\begin{cases}
dv(z, t)/dt = z^4 [J, v(z, t)], \\
v(z_j, t)/dt = z_j^4 [J, v(z_j, t)],
\end{cases}
\]

then the associated potential \( Q(x, t) \) satisfies \( Q_t = [J, G_4] \), where \( G_4 \) is the coefficient of \( \frac{1}{x} \) of the asymptotic expansion of \( MJM^{-1} \) for \( x \to \infty \). If \( v(z, 0), v(z_j, 0) \) satisfy the constraints (3.1) and \( v(z, t), v(z_j, t) \) evolve as (3.2), then \( v(z, t), v(z_j, t) \) also satisfy the constraints (3.1). Hence the inverse problem is solvable for all time \( t \). The associated potential \( (Q, P) \) is of the form \( Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix} \), \( P = Q(adJ)^{-1}Q \), and \( q \) satisfies

\[
q_t = (i/2)q_{xx} + (i/2)q^2 q_x + q |q|^4.
\]

Equation (4.3) can be transformed into the DNLS explicitly. Let \( u = q \exp(\int_{-\infty}^x -iqq^*) \), the \( u \) satisfy the derivative nonlinear Schrödinger equation (DNLS) \( u_t = iu_{xx} - (u^2 \cdot u^*)_x \), which was considered by Kaup and Newell [12] and Gerzhikov et al. [11]. The global existence (in time \( t \)) of the Schwartz class solutions of Eq. (3.3) was also obtained for the generic initial data by Lee via the \( L^2 \)-estimate [19].

Now, given the scattering data \( \{v(z, 0) \equiv I, z_1, z_2, \ldots, z_N, -z_1, -z_2, \ldots, -z_N, \bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N, -\bar{z}_1, -\bar{z}_2, \ldots, \ldots, -\bar{z}_N, v(z_j, 0)\} \) and supposing that \( v(z_j, t) \) evolve as (3.2), the inverse problem amounts to solving a linear system. Now we derive the formula of the solitons.

(I) 1-Soliton Solution: We look for

\[
M = I + \frac{A_0}{z - z_0} + \frac{C_0}{z + z_0} + \frac{B_0}{z - z_0^*} + \frac{D_0}{z + z_0},
\]