UNSTEADY INCOMPRESSIBLE FLUID FLOW PAST A CASCADE OF THIN CURVED PLATES

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A large number of papers have been devoted to the study of unsteady flow past airfoil cascades. The majority of authors solve the problem for slightly curved profiles oscillating at low angles of attack.

Among other work, we note that of Søhngen [1] on the flow past a dense cascade of plates oscillating synchronously and in phase in a potential fluid flow at a high angle of attack. Samoilovich [2] studied the flow past a cascade of plates of arbitrary shape oscillating with an arbitrary phase shift between neighboring plates. He presents the solution for the case of variable circulation in the quasisteady formulation. Stepanov [3] studied the same question with a linear approach to the flow behind the cascade. Musatov [4] examined the problem of the flow past a cascade of plates oscillating with an arbitrary phase shift between neighboring plates in a fluid flow, again at a high angle of attack, and considered the variation of the relative position of the plates during the oscillation process.

The present paper considers the flow of a perfect incompressible fluid past a cascade of thin curved oscillating plates with account for the relative displacements of the plates during oscillation. To determine the intensity of the bound vortices per unit length, a linear integral equation is obtained. This represents a generalization of the Bimbaum equation to the case considered (see [5]). Equations are presented for calculating the unsteady aerodynamic forces and moments acting on the plates. As an example, the aerodynamic forces and moments are calculated for the quasistationary formulation of the problem.

Let us consider in the complex plane \( z = x + iy \), a cascade of thin curved plates (Fig. 1) executing steady-state harmonic oscillations in a perfect incompressible fluid flow. The \( x, y \) coordinate system is tied to the average position of the initial reference plate. The coordinate origin is located at the leading edge of the reference plate, and the \( x \) axis is directed downstream along the chord. The reference plate is described by the arc

\[ y = Y(x) \quad (0 \leq x \leq b), \]

where \( b \) is the chord length.

The plates oscillate with small amplitudes, synchronously, and with the arbitrary phase shift \( \varphi_k (|m| \leq 1) \) between neighboring plates. We will assume that outside the cascade, and away from the vortex trails behind each plate the fluid flow is potential. The velocity potential in the \( x, y \) coordinate system satisfies the Laplace equation and the following boundary conditions.

1. The condition of unseparated flow on the plates.
2. The Zhukovskii-Kutta condition of finite velocity at the trailing edges of the plates.
3. The condition of damping of the unsteady velocity disturbances at an infinite distance ahead of the cascade.

We replace each plate by a continuous vortex sheet, for which the unknown intensity per unit length is represented in the form

\[ \Gamma_k^s (s, t) = \Gamma_0 (s) + \Gamma (s) e^{i \omega (t + \kappa t)}, \quad (1.1) \]

\[ \Gamma (s) = \Gamma^* (s) + i \Gamma^* (s). \]

Here \( \Gamma_0 (s) \) is the vortex intensity per unit length in the steady flow; \( s \) is the plate arc length, measured from the leading edge; \( k = 0, \pm 1, \pm 2, \ldots \) is the plate number in the cascade; \( \kappa \) is the imaginary unit, serving to designate only time-dependent processes.

In accordance with the Thomson theorem, the variation of the vortex intensity on the plate by an amount \( \Delta \Gamma \) gives rise to the appearance in the fluid of a free vortex with intensity \( -\Delta \Gamma \). Following the scheme of Birnbaum [5], we will assume that free vortices continuously appear at every point on the plate and move relatively to the plate with velocity \( W_0 (s) \), equal to the average of the limiting fluid velocities on approach to the plate from above and below in the steady flow. On reaching the trailing edge, the free vortices continue to move with the velocity of the steady stream and form a vortex trail which coincides with the streamline in the steady stream. The intensity of the sheet of free vortices \( \Gamma_k (s, t) \) is equal to the sum of the intensities of the free vortices that form at the instant of time \( t_1 \leq t \) at the points \( s_1 \leq s \) and which by the time \( t \) have reached the point \( s \) of the \( k \)-th profile.

Thus,

\[ H_k (s, t) = \int_{s_1}^{s} \frac{1}{W_0 v (s_1)} \left[ \frac{\partial \Gamma_k^s (s_1, t)}{\partial t} \right] \, ds_1, \]

\[ t_1 = t - \int_{s_1}^{s} \frac{d\theta}{W_0 v (s_1)}, \]

and the integration is performed along the profile arc.

With account for (1.1), we obtain respectively for the free vortices on the \( k \)-th plate and in the wake of the \( k \)-th plate

\[ H_k (s, t) = H_k (s_0, t) e^{-i \omega T(s, t)} \]

\[ T (s_1, s) = \int_{s_1}^{s} \frac{d\theta}{W_0 v (s_1)}, \quad (1.2) \]

\[ H_k (s, t) = h_k (s_0, t) e^{-i \omega T (s, t)} \]

\[ T (0, s) = \int_{s}^{0} \frac{d\theta}{W_0 v (s)}, \quad (1.3) \]

Here \( s_0 \) is the total length of the profile arc; \( \sigma \) is the arc length of the line coinciding with the vortex trail and measured from the trailing edge of the plate; \( W_0 (s) \) is the cross-flow velocity of the free vortices in the wake.
Let \( W_n(x, y, t) \) be the normal component of the disturbed fluid velocity at the point \((x, y)\) of the reference plate; \( u_n(x, y, t) \) is the normal component of the oscillation velocity at the point \((x, y)\) of the reference plate. The condition of no flow through the surface of the reference plate may be written in the form

\[
W_n(x, y, t) - u_n(x, y, t) = 0. \tag{1.4}
\]

The disturbed fluid velocity at the point \((x, y)\) at time \(t\) is equal to

\[
W(x, y, t) = V_0 + v(x, y, t) + v_1(x, y, t) + v_2(x, y, t). \tag{1.5}
\]

Here \( V_0 \) is the average fluid velocity, equal to the average of the velocities infinitely far ahead of and behind the cascade in the steady flow; \( v \) is the fluid velocity induced by the bound vortices; \( v_1, v_2 \), are, respectively, the fluid velocities induced at the plates by the free vortices and the vortex trails.

In determining the induced velocities of the fluid, we will take account of the relative displacement of the plates in the oscillation process. Limiting ourselves to the linear formulation of the problem, we will then retain only terms of no higher than first order of smallness relative to the amplitude of the plate oscillations.

Here \( \Delta z \) denotes the displacement in the oscillation process of the point \((x, y(x))\) of the reference plate. For the case of an arbitrary small plate deformation, following a harmonic law we have

\[
\Delta z = \delta(t) F(x), \quad \delta(t) = \delta_0 e^{i\omega t} \quad (|\delta_0| \ll 1). \tag{2.1}
\]

Here \( F(x) = F'(x) + iF''(x) \) is a function which characterizes the mode of oscillation of the plate; \( \delta_0 \) is the dimensionless amplitude of the oscillations.

We will term those points which lie on a single straight line parallel to the cascade front when each plate is in the central position corresponding points. The complex coordinates of the corresponding points of the plates at time \(t\) are equal to

\[
Z(t) = z_0 + ikhe^{-i\theta} + \Delta z e^{i\kappa m} \quad (k = 0, \pm 1, \pm 2, \ldots) \tag{2.2}
\]

Here \( h \) is the cascade pitch, \( \beta \) is the stagger angle, \( z_0 \) is the complex coordinate of a point of the reference plate in its central position.

The infinite series of elementary vortices \( \Gamma_k * ds \), located at corresponding points of the plates \( \xi_k \), induce at the point \( z \) the complex-conjugate velocity

\[
d(v_x - i v_y) = \frac{1}{2\pi i} \sum_{k=-\infty}^{+\infty} \Gamma_k e^{i\kappa m} \frac{dz}{z - z_0 - ikhe^{-i\theta}} ds.
\]

Taking account of (2.1) and (2.2), we obtain the expression for the velocity induced at a point of the reference plate by the bound vortices of all the plates

\[
v_x - iv_y = \frac{1}{2\pi i} \sum_{k=-\infty}^{+\infty} \frac{\Gamma_k e^{i\kappa m}}{z_0 - z - ikhe^{-i\theta}} (z_0 - z_0 - ikhe^{-i\theta}) ds
\]

Here, the integration is performed along the profile arc

\[
\xi_0 = \xi + iY(\xi).
\]

Correct to small terms of the first order in the amplitude of the oscillations, we represent the integrand in the form

\[
\Gamma_0(e) \sum_{m=-\infty}^{+\infty} \frac{1}{z_0 - z - ikhe^{-i\theta}} - \Gamma_0(e) e^{i\omega t} \sum_{m=-\infty}^{+\infty} \frac{1}{z_0 - z - ikhe^{-i\theta}}
\]

\[
- \Gamma_0(e) \delta e^{i\omega t} \left\{ F(x) \sum_{m=-\infty}^{+\infty} \frac{1}{z_0 - z - ikhe^{-i\theta}} - F(x) \right\}
\]

The latter expression may be transformed and summed:

\[
\Gamma_0(e) \frac{\kappa}{b} \left[ \left( \frac{z_0 - z_0 - \frac{\kappa}{b}}{b} \right)^{-1} + \sum_{m=-\infty}^{+\infty} \left( \frac{z_0 - z_0 - \frac{\kappa}{b}}{b} - ik\pi \right)^{-1} \right] +
\]

\[
+ \Gamma_0(e) e^{i\omega t} \frac{\kappa}{b} \left[ \left( \frac{z_0 - z_0 - \frac{\kappa}{b}}{b} \right)^{-1} + \sum_{m=-\infty}^{+\infty} \left( \frac{z_0 - z_0 - \frac{\kappa}{b}}{b} - ik\pi \right)^{2} \right]
\]

\[
- \Gamma_0(e) \delta e^{i\omega t} \frac{\kappa}{b} \left[ F(x) e^{i\omega t} + \sum_{m=-\infty}^{+\infty} \left( \frac{z_0 - z_0 - \frac{\kappa}{b}}{b} - ik\pi \right)^{-2} \right]
\]

\[
- \frac{F(x)}{b} \left[ \left( \frac{z_0 - z_0 - \frac{\kappa}{b}}{b} \right)^{-2} + \sum_{m=-\infty}^{+\infty} \left( \frac{z_0 - z_0 - \frac{\kappa}{b}}{b} - ik\pi \right)^{-2} \right] =
\]

\[
= \frac{\pi i}{b} \Gamma_0(e) R_0(z_0, \xi_0) + \frac{\pi i}{b} \Gamma_0(e) e^{i\omega t} R(m, z_0, \xi_0) -
\]

\[
- \frac{\pi i}{b} \delta e^{i\omega t} \Gamma_0(e) \left[ F(x) - \frac{F(x)}{b} \right] \beta_{1}(z_0, \xi_0) - \frac{F(x)}{b} \beta_{2}(m, z_0, \xi_0).
\]

Here

\[
R(m, z_0, \xi_0) = \frac{\pi}{\alpha_i} \left[ \left( \frac{m}{m} - m \right) \times \left( \frac{z_0 - z_0}{b} \right) \right]
\]

\[
- \frac{\pi i}{\alpha_i} \left[ \frac{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) }{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) } \right] \csc \left( \frac{z_0 - z_0}{b} \right)
\]

\[
\alpha = \frac{\pi b}{h} e^{i\theta}, \quad R_0(z_0, \xi_0) = \frac{\pi}{\alpha_i} \left[ \frac{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) }{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) } \right] \csc \left( \frac{z_0 - z_0}{b} \right)
\]

\[
\beta_{1}(z_0, \xi_0) =
\]

\[
= \frac{\pi i}{\alpha_i} \left[ \frac{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) }{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) } \right] \csc \left( \frac{z_0 - z_0}{b} \right)
\]

\[
\beta_{2}(m, z_0, \xi_0) =
\]

\[
= \frac{\pi i}{\alpha_i} \left[ \frac{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) }{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) } \right] \csc \left( \frac{z_0 - z_0}{b} \right)
\]

\[
= \frac{\pi i}{\alpha_i} \left[ \frac{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) }{\times \left( \frac{z_0 - z_0}{b} \right) \csc \left( \frac{z_0 - z_0}{b} \right) } \right] \csc \left( \frac{z_0 - z_0}{b} \right)
\]