DARBOUX TRANSFORMATIONS FOR NON-ABELIAN AND NONLOCAL EQUATIONS OF THE TODA CHAIN TYPE

M. A. Sall'

Darboux transformations are used to construct explicit solutions for the two-dimensionalized Toda chain, sine-Gordon equations and their non-Abelian analogs, the nonlinear Schrödinger equation, the nonlocal Toda equation, and non-Abelian equations of Langmuir oscillations.

1. Introduction

Darboux transformation as a method of simultaneous finding of a potential and the eigenfunctions of the Schrödinger equation on an axis corresponding to it was proposed at the end of the last century [1]. Renewal of interest in the method is due to Wahlquist's paper [2], which in fact demonstrated the identity of the Bäcklund transformation for the Korteweg–de Vries (KdV) equation and the Darboux transformation for the corresponding spectral problem.

These ideas were developed further by Matveev [3-6] and Matveev and the present author [7-9].

Darboux transformations are the simplest way of obtaining a large class of solutions of nonlinear equations. They make it possible to describe in a unified manner many-soliton, rational, periodic, and other solutions on an arbitrary background without requiring full implementation of the inverse problem method.

In the present paper, we describe the application of these transformations to the non-Abelian Toda chain, which was proposed by Polyakov and investigated by Mikhailov [10] and Krichever (Appendix to [11]). In addition, we consider the equations of Langmuir oscillations, the sine–Gordon equation, the nonlinear Schrödinger equation, and the decoupled nonlinear Schrödinger equation. A review of the existing results can be found in the book [12]. For the equations of Langmuir oscillations and the sine–Gordon equation we consider their non-Abelian analogs. We follow the connections between the non-Abelian analog of the equations of Langmuir oscillations, the equations of nonlinear filters, and the discrete analog of the nonlinear Schrödinger equation [13, 14]. We also investigate an equation which we call the "nonlocal Toda equation," which is a natural nonlocal analog of the finite chains considered by Leznov, Saveliev, and Smirnov [15, 16]. For the nonlocal Toda equation, we construct hyperelliptic solutions.

2. Darboux Transformation

The method of Darboux transformations is based on the possibility of representing a nonlinear equation for the function $u(x, t)$ in the form of the consistency condition of a system of linear equations:

$$\begin{cases}
\psi_+ = L(u)\psi_-
\
\psi_- = A(u)\psi_+
\end{cases}$$

where $A(u)$ and $L(u)$ are linear operators.

Suppose there exists a linear transformation of the eigenfunctions

$$\tilde{\psi} = M\psi,$$

such that Eqs. (1) are invariant with respect to it, i.e., are carried into themselves apart from the replacement of $u$ by $\tilde{u}$. In this case, $\tilde{u}$ is also a solution of the original equation, and we call the transformation (2) a Darboux transformation.

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In Table I, we give the nonlinear equations (I, Eqs. (3)), the linear systems whose consistency conditions the given equations are (II, Eqs. (4)), the Darboux transformations for the solutions of these systems (III, Eqs. (5)), and the transformations of the solutions of the original equations induced by them (IV, Eqs. (6)).

In Table I, we include the following equations.

1. The non-Abelian Toda chain.
2. The non-Abelian analog of the equations of Langmuir oscillations.
3. The non-Abelian analog of the sine-Gordon equation.

In all these equations, $g_0$, $v$, $u$, $a$, $v$, $w$ are invertible matrices of order $N$, $h$ is a matrix spectral parameter, $\psi$ and $\psi$ are matrix-valued solutions of the linear system, and $g_0$ and $\psi_0$ are solutions of the linear system corresponding to the fixed value $k = k_0$.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>((g_n h_n)^{-1} + g_{n+1} g_{n+1}^{-1} - g_n g_n^{-1})</td>
<td>(\psi_{n+1} = u_n \psi_n - \psi_{n+1})</td>
<td>(\bar{\psi}_n = \psi_n - \sigma_n \psi_n^{-1})</td>
<td>(\bar{\psi}_n = \sigma_n \psi_n^{-1})</td>
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<td>2</td>
<td>(a_{n+1} = a_n - a_{n+1} a_n a_{n+1})</td>
<td>(\psi_{n+1} = \psi_n - \sigma_n \psi_n^{-1})</td>
<td>(\bar{\psi}_n = \psi_n - \sigma_n \psi_n^{-1})</td>
<td>(\bar{\psi}_n = \sigma_n \psi_n^{-1})</td>
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<tr>
<td>3</td>
<td>(\psi_{n+1} = \psi_{n+1} - \sigma_{n+1} \psi_{n+1})</td>
<td>(\psi_{n+1} = \psi_n - \sigma_n \psi_n^{-1})</td>
<td>(\bar{\psi}_n = \sigma_n \psi_n^{-1})</td>
<td>(\bar{\psi}_n = \sigma_n \psi_n^{-1})</td>
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<td>4</td>
<td>(a_{1} - a_{3} = 4 (T + T^{-1} - 2) e^{ \alpha})</td>
<td>(\psi_{x} = \alpha e^{\gamma} \psi_{y} + e^{\gamma} + \frac{1}{\alpha} T - \psi_{y} + \psi_{y})</td>
<td>(\bar{\psi} = \psi_{y} - \alpha T \psi_{y})</td>
<td>(\bar{\psi} = \psi_{y} - \alpha T \psi_{y})</td>
</tr>
<tr>
<td>5</td>
<td>(\psi_{x} = \alpha \psi_{y} + e^{\gamma} + \frac{1}{\alpha} T - \psi_{y} + \psi_{y})</td>
<td>(\psi_{x} = \alpha \psi_{y} + e^{\gamma} + \frac{1}{\alpha} T - \psi_{y} + \psi_{y})</td>
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