Generalized Master Equations for Continuous-Time Random Walks

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An equivalence is established between generalized master equations and continuous-time random walks by means of an explicit relationship between $\psi(t)$, which is the pausing time distribution in the theory of continuous-time random walks, and $\phi(t)$, which represents the memory in the kernel of a generalized master equation. The result of Bedeaux, Lakatos-Lindenburg, and Shuler concerning the equivalence of the Markovian master equation and a continuous-time random walk with an exponential distribution for $\psi(t)$ is recovered immediately. Some explicit examples of $\phi(t)$ and $\psi(t)$ are also presented, including one which leads to the equation of telegraphy.

KEY WORDS: Generalized master equations; random walks; statistical mechanics; transport theory.

1. INTRODUCTION

A standard starting point for the discussion of various random walks and other transport processes is the master equation

$$\frac{d\tilde{P}(l, t)}{dt} = -\alpha\tilde{P}(l, t) + \alpha \sum_{l' \neq l} p(l, l') \tilde{P}(l', t)$$

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1 For continuity, the reader is directed to the article entitled “Random Walks on Lattices. IV. Continuous Time Walks and Influence of Absorbing Boundaries,” by E. W. Montroll and H. Scher, which will appear in Volume 9, Number 2, of this journal, and which should precede the following article. Regrettably, the two articles were inadvertently switched during processing.

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where \( \bar{P}(l, t) \) is the probability that a system of interest is in state \( l \) at time \( t \) and \( \alpha p(l, l') \) is the probability per unit time of a transition from \( l' \) to \( l \). Equation (1) is of course equivalent to the "gain-loss" form of the master equation

\[
d\bar{P}(l, t)/dt = \alpha \sum_{l'} [p(l, l') \bar{P}(l', t) - p(l', l) \bar{P}(l, t)]
\]

since transition probabilities have the normalization

\[
\sum_{l'} p(l', l) = 1
\]

We now interpret the states \{\( l \)\} to be lattice points on a periodic space lattice and the system to be a random walker on the lattice. This interpretation is not necessary but it is made to give a direct contact with the results of Ref. 1. It was emphasized there that certain interesting random walks cannot be described by (1).

The basic quantity employed in the preceding paper is the pausing time distribution function \( \psi(t) \) (the probability density function for the time \( t \) between the arrival of a walker at a given lattice point and the initiation of the next step to another site). All lattice points were postulated to be equivalent (periodic boundary conditions being used) so that \( \psi(t) \) can be taken to be universal for all points. The methods of the preceding paper involve the random walk generating function which satisfies the Green’s function equation [with \( \bar{P}(l, 0) = \delta_{l,0} \)]

\[
G(l, z) - z \sum_{l'} p(l - l') G(l', z) = \delta_{l,0}
\]

The form for \( G(l, z) \) on a \( d \)-dimensional periodic lattice with \( N \times N \times N \times \cdots \) lattice points in each direction (with periodic boundary conditions) is

\[
G(l, z) = N^{-d} \sum_{(s_j = 1)}^{N} \sum_{s_j=1}^{N} e^{ik.l}/[1 - z\lambda(k)]
\]

where \( k_j = 2\pi s_j/N \) and \( \lambda(k) \) is the so-called structure function

\[
\lambda(k) = \sum_{l} p(l) e^{ik.l}
\]

The quantity \( \bar{P}(l, t) \) was shown to be related to \( G(l, z) \) through the inverse Laplace transform formula\(^{(1,2)}\)

\[
P(l, t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{ut} du \{[1 - \psi^*(u)]/u\} G(l, \psi^*(u))
\]

where \( \psi^*(u) \) is the Laplace transform of \( \psi(t) \).