NLS-TYPE EQUATIONS IN 2+1 DIMENSIONS: NEW TYPE
OF SOLUTIONS AND NON-ISOSPECTRAL PROBLEMS

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Integrable systems of NLS type in 2+1 dimensions are studied in three ways: their special solutions by Lie point symmetries, 'breaking'-solutions by the ISM, as well as the non-isospectral problems.

1. INTRODUCTION

Among the nonlinear Schrödinger (NLS)-type equations in 2+1 dimensions, the DS-1 equations [1]

\[ \begin{align*}
  iq_t + q_{xx} + q_{yy} + (u + v)q &= 0, \\
  ur_t + r_{xx} + r_{yy} + (u + v)r &= 0
\end{align*} \]

(1)

with $\xi = x + y$ and $\eta = x - y$ are of the primary concern. The solution of (1) by the inverse scattering methods (ISM) can be found in [1, 2]. Another type of (2+1)-dimensional generalisation [3, 4] of the NLS equation reads

\[ \begin{align*}
  iq_t &= q_{xy} - 2q \partial_x^{-1} \partial_y(qr), \\
  ir_t &= -r_{xy} + 2r \partial_x^{-1} \partial_y(qr)
\end{align*} \]

(2)

with $\partial_x^{-1} = \int_{-\infty}^{x} \partial_x$ and $r = -q^*$. Equations (2) are found to pass the Painlevé test and possess infinitely many conservation laws [5, 3]. Our main objective of this work, therefore, is to show that: (i) the successive use of Lie symmetries in two different sequences to DS-1 equations (1) may lead to the same ordinary differential equation (ODE), which therefore induces two different sets of similarity solutions [6]; (ii) a new type of solutions, sometimes called the break solitons, can be derived for (2) from an ISM; (iii) a non-isospectral generalisation of (1) will transform the time evolution of the inverse data into a linear integral equation [7].

2. SOME SPECIAL SIMILARITY REDUCTIONS OF THE
DS-1 EQUATIONS

Our purpose here is to show that two different ways of consecutive similarity reductions for the DS-1 equations may lead to a single same ODE. In other words, this same ODE may generate different classes of similarity solutions for the DS-1 equations.

First, we note that Eqs. (1) admit the following Lie point symmetry $\Xi = \mu_1 \partial_\xi + \mu_2 \partial_\eta + \mu_3 \partial_t + \nu_1 \partial_q + \nu_2 \partial_r + \nu_3 \partial_u + \nu_4 \partial_v$, where

\[ \begin{align*}
  \mu_1 &= \xi c'(t) + 2a(t), \\
  \mu_2 &= \eta c'(t) + 2b(t), \\
  \mu_3 &= 2c(t), \\
  \nu_1 &= q \left\{ i\left[ \xi a'(t) + \eta b'(t) \right] + i c'(t)(\xi^2 + \eta^2)/4 - c'(t) + d(t) \right\}, \\
  \nu_2 &= r \left\{ -i\left[ \xi a'(t) + \eta b'(t) \right] - i c''(t)(\xi^2 + \eta^2)/4 - c'(t) - d(t) \right\}, \\
  \nu_3 &= -2u c'(t) + \eta b''(t) + \eta^2 c'''(t)/4 - id'(t)/2 + e(t) \\
  \nu_4 &= -2v c'(t) + \xi a''(t) + \xi^2 c'''(t)/4 - id'(t)/2 - e(t)
\end{align*} \]

(3)

where $a, b, \ldots, e$ are arbitrary (smooth) functions of time $t$. Certain Lie symmetries for DS equations are also considered by, e.g., Winternitz [8]. We note that $\Xi$ is a Lie symmetry of (1) means that Eqs. (1) are invariant [9] under the Lie group generated by the symmetry $\Xi$. By choosing, in particular, $a = b = t$, $c = \frac{1}{2}$, and $d = e = 0$, we reduce (1) to

\[ \begin{align*}
  Q_{XX} + Q_{TT} + \frac{1}{2} Q(U + V + iT) &= 0, \\
  U_X + U_T - \frac{1}{2} (QR)_X + \frac{1}{2} (QR)_T &= 0, \\
  R_{XX} + R_{TT} + \frac{1}{2} R(U + V + iT) &= 0, \\
  -V_X + V_T + \frac{1}{2} (QR)_X + \frac{1}{2} (QR)_T &= 0
\end{align*} \]

(4)
via the solutions of $d\xi/\mu_1 = d\eta/\mu_2 = dt/\mu_3 = dq/\nu_1 = dr/\nu_2 = du/\nu_3 = dv/\nu_4$. To be more precise, Eqs. (4) are reduced from (1) by the transformation

$$
X = \xi - \eta, \quad T = \xi + \eta + 2it^2, \quad u = U(X,T), \quad v = V(X,T),
$$

$$
g = Q(X,T) \exp \left\{ Tt - \frac{2}{3} it^3 \right\}, \quad r = R(X,T) \exp \left\{ -Tt + \frac{2}{3} it^3 \right\}.
$$

Obviously Eqs. (4) remain a set of nonlinear partial differential equations (PDEs). Therefore, we shall apply again the above procedure to the newly obtained system (4). The general Lie point symmetry $\Xi = \mu_1 \partial_X + \mu_2 \partial_T + \nu_1 \partial_Q + \nu_2 \partial_R + \nu_3 \partial_U + \nu_4 \partial_V$ for (4) is thus found to be given by

$$
\begin{align*}
\mu_1 &= \dot{a}X + \dot{b}, & \mu_2 &= \dot{a}T + \dot{c}, & \nu_1 &= -Q(\dot{a} + \dot{d}), & \nu_2 &= -R(\dot{a} - \dot{d}), \\
\nu_3 &= -2\dot{a}U + \frac{3}{2} \dot{a}i(X - T) - \frac{1}{2} i\dot{e} + \dot{e}, & \nu_4 &= -2\dot{a}V - \frac{3}{2} \dot{a}i(X + T) - \frac{1}{2} i\dot{e} + \dot{e}.
\end{align*}
$$

By taking $\dot{a} = 1$ and $\dot{b} = \dot{c} = \dot{d} = \dot{e} = 0$, for instance, we obtain from the linear PDE

$$
\begin{align*}
\frac{dX}{X} &= \frac{dT}{T} = \frac{dQ}{-Q} = \frac{dR}{-R} = \frac{dU}{-2U + \frac{3}{2} i(X - T)} = \frac{dV}{-2V - \frac{3}{2} i(X + T)}
\end{align*}
$$

the following similarity transformation:

$$
\zeta = X/T, \quad Q = f(\zeta)/T, \quad R = g(\zeta)/T, \quad U = \dot{u}(\zeta)/T^2 - i(1 - \zeta)/T^2, \quad V = \dot{v}(\zeta)/T^2 - i(1 + \zeta)/T^2.
$$

The new transformation (6) will then reduce the PDEs (4) into a set of ODEs

$$
\begin{align*}
2(\zeta^2 + 1)f'' + 8\zeta f' + (4 + \dot{u} + \dot{v})f &= 0, & 2(\zeta - 1)f' + (\zeta + 1)(f')' + 4\dot{u} + 2fg &= 0, \\
2(\zeta^2 + 1)g'' + 8\zeta g' + (4 + \dot{u} + \dot{v})g &= 0, & 2(\zeta + 1)f' + (\zeta - 1)(f')' + 4\dot{v} + 2fg &= 0,
\end{align*}
$$

where $'$ denotes the differentiation with respect to the variable $\zeta$ in this section. In order to reduce Eqs. (7) even further, we set $f = g, \quad \dot{u} = (\varphi + \psi)/2, \quad \dot{v} = (\varphi - \psi)/2$, whose compatibility can be easily verified from the above Eqs. (7). A surprising property of (7) is that we can actually solve $\varphi$ and $\psi$ explicitly in terms of a single function $f(\zeta)$. This solution can be shown to be

$$
\begin{align*}
\varphi &= -\left\{ \frac{d}{d\zeta} \left[ 2(\zeta^2 + 1)f' + 4\zeta f \right] \right\}/f(\zeta), & \psi &= -\frac{1}{2} \left\{ \frac{d}{d\zeta} \left[ (\zeta^2 - 1)\varphi + (\zeta^2 + 1)f^2 \right] \right\},
\end{align*}
$$

where $f(\zeta)$ is determined by the following single ODE:

$$
\begin{align*}
(\zeta^4 - 1)(f^2 f^{(4)} - 2f' f^{(3)} + f^{(2)} f'' - 2f' f^{(2)} f'' - f^{(1)} f^{(2)} + f^{(1)} f^{(2)} f') - (\zeta^2 + 1)(f^4 f'' + f^2 f^2) \\
- 8(4\zeta^2 - 1)ff^{(2)} - 2(11\zeta^2 - 5)f f' f'' + 8\zeta(\zeta^2 - 1)f^{(3)} + 2\zeta(7\zeta^2 - 1)f^2 f^{(3)} \\
+ 4(13\zeta^2 - 1)f^2 f'' - 4\zeta f' f'(f' - 12) - f^3(3f^2 + 2ff' - 12) &= 0.
\end{align*}
$$

In deriving (8) and (9), Eqs. (7) are first transformed into

$$
\begin{align*}
2\psi + [(\zeta^2 - 1)\varphi + (\zeta^2 + 1)f^2]'' &= 0, & -\psi' + \zeta\varphi' + 2\varphi + \zeta(f^2)' + 2f^2 &= 0, \\
\varphi f + [2(\zeta^2 + 1)f' + 4\zeta f]' &= 0
\end{align*}
$$

and then into

$$
\begin{align*}
[(\zeta^2 - 1)\varphi + (\zeta^2 + 1)f^2]'' + 2\zeta\varphi' + 4\varphi + 2(f^2)' + 4f^2 &= 0.
\end{align*}
$$

before being finally reduced to (8). We note that ODEs obtained from integrable NEEs are known [10] to exhibit good behaviours. Obviously each solution of Eq. (9) then gives automatically via (8), (6), and (5) a similarity solution for the original equations (1). In other words, we have first reduced (1) by a symmetry in (3) to a PDE system of fewer independent variables and then reduced by another symmetry the newly derived system furthermore to a single ODE. Obviously we may also make another choice of the parameters in the Lie symmetry (3) for the first step of the reduction of Eqs. (1). In fact, we found in this way that a different procedure of reductions of (1) which may also give rise to (9) is given by

$$
T = \xi - \eta, \quad T = \xi + \eta - 2t^2, \quad \zeta = X/T,
$$

$$
g = f(\zeta) \exp \left\{ iTt + \frac{2}{3} it^3 \right\}/T, \quad u = \left[ \varphi(\zeta) + \psi(\zeta) \right]/(2T^2) + \frac{1}{2}(1 - \zeta)/T,
$$

$$
r = f(\zeta) \exp \left\{ -iTt - \frac{2}{3} it^3 \right\}/T, \quad v = \left[ \varphi(\zeta) - \psi(\zeta) \right]/(2T^2) + \frac{1}{2}(1 + \zeta)/T,
$$

where $\varphi$ and $\psi$ are also determined via (8) by the ODE (9). It seems quite interesting that the two different ways of consecutive similarity reductions should reduce the DS-1 equations into the same single ODE.