CLIPPING – A NEW INVESTIGATION METHOD FOR PDES IN COMPACT DOMAINS

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1. INTRODUCTION

We consider a class of nonlinear evolution equations with dispersive linear part and nonlinear part small enough in the sense that the nonlinear effects are weak. We also hold that these equations admit resonant interactions.

Let us write out an equation of this class in the form

\[ L(\psi) = \varepsilon N(\psi), \]

where \( L(\psi) \) is the linear part of the equation and \( L(\omega) = 0 \) for dispersion \( \omega \), \( N(\psi) \) is the nonlinear part of the equation, and \( 0 < \varepsilon \ll 1 \) is a small parameter. The resonant conditions for \( s \)-wave interactions have the form

\[ \omega(k_1) \pm \omega(k_2) \pm \cdots \pm \omega(k_s) = 0, \]
\[ k_1 \pm \cdots \pm k_s = 0, \]

where \( k_i \) are wave vectors, \( i = 1, \ldots, s \).

The study of system (2) in compact domains with zero or periodic boundary conditions leads to the necessity of solving it in integers. We have found [1] that these systems have a few common features, namely:

- waves interacting resonantly can be partitioned into disjoint classes which do not cross-interact; there is no energy flow between these classes, so it is possible to study each class independently; the number of elements in these classes is not large (as a rule, a class contains 3 to 5 waves);

- most waves do not participate in any resonant interactions at all (for some specific forms of dispersion relation this amounts to 60 to 80 %);

- the number of resonantly interacting waves strongly depends on the shape of the basin (for example, on the ratio of its sides for a rectangular one); for some specific forms of dispersion relation a set of basins can be described where no waves interact resonantly;

- interaction is local in the sense that for any fixed wave its interaction domain (i.e., the spectral region which contains all the waves which can interact with the given one) is bounded and can be written out effectively.

The present paper is devoted to a detailed study of the first feature – the partitioning of the resonantly interacting waves into small groups. We demonstrate that this feature gives us a powerful tool for studying the energy transfer via the spectrum governed by some specific PDE for a wide class of initial conditions. In Sec. 2, the value of the frequency imbalance is estimated for which the partition is still conserved. The method which allows us to clip out the largest part of the spectral modes and use only a few modes to describe the wavefield evolution governed by a specific PDE – the clipping method – is described in Sec. 3. The results of applying the clipping method to the barotropic vorticity equation are presented in Sec. 4. We briefly discuss the obtained results in Sec. 5.

2. FREQUENCY IMBALANCE

The partition has been obtained as the property of the exact solutions of Eq. (2). To apply these results to some physical problem, first we have to answer the following question. Does there exist some non-zero resonance width, i.e.,
\[ \Omega = \omega(k_1) \pm \omega(k_2) \pm \cdots \pm \omega(k_s) > 0, \]

such that the properties formulated in Sec. 1 are still valid?

The estimation of \( \Omega \) as a linear form on the values of the function \( \omega \) taken in different points of its domain of definition depends on the \( \omega \) changing domain. Let us consider a few cases.

1. Let \( \omega : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q} \), where \( \mathbb{Z} \) denotes the set of integers and \( \mathbb{Q} \) denotes the set of rational numbers (example: \( \omega = -2m/n(n+1) \) for spherical planetary waves, \( m \) and \( n \) are wave numbers). Then, obviously, \( \Omega \) can be represented as the difference of two rational numbers \( \frac{c}{b} \) and \( \frac{a}{d} \), and we have trivial estimation
\[
|\Omega| = \left| \frac{a}{b} - \frac{c}{d} \right| > \frac{1}{bd} > 0. \tag{5}
\]

2. Let \( \omega : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Q}(\alpha) \), where \( \alpha \) is an algebraic number, i.e., it is a zero of some polynomial \( P(x) = a_0 x^r + a_1 x^{r-1} + \cdots + a_r \), where \( a_i \) denote integers, not all zero. The field \( \mathbb{Q}(\alpha) \) denotes the algebraic expansion of \( \mathbb{Q} \), i.e., the set of number of the form \( \frac{a}{b} + \frac{c}{d} \alpha \) with \( a, b, c, d \in \mathbb{Q} \). (Example: \( \omega^2 = k^3 \) for capillary waves, \( k \) is the modulus of the wave vector.) In this case, we may use the generalization of the Thue–Siegel–Roth theorem [2]:

If the algebraic numbers \( \alpha_1, \alpha_2, \ldots, \alpha_s \) are linearly independent with 1 over \( \mathbb{Q} \), then for any \( \varepsilon > 0 \) we have
\[
|q_1 \alpha_1 + q_2 \alpha_2 + \cdots + q_s \alpha_s - p| > c q^{-\varepsilon}, \quad \forall p, q_1, \ldots, q_s \in \mathbb{Z} \quad \text{with} \quad q = \max_i |q_i|. \tag{6}
\]

The constant \( c \) has to be constructed for every specific set of algebraic numbers separately.

3. Let \( \omega : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{R} \), where \( \mathbb{R} \) denotes the set of real numbers; \( \omega \) is an arbitrary real-value function of integer variables. To find the frequency imbalance \( \Omega \) in the finite spectral domain \( D, D = \{(m, n) : 0 < m, n \leq T < \infty\} \), it is enough to calculate it as
\[
\Omega = \min_p (\Omega_p), \tag{8}
\]

where
\[
\Omega_p = \omega(k_1^p) \pm \omega(k_2^p) \pm \cdots \pm \omega(k_s^p), \tag{9}
\]

and \( p \) is finite because the total number of wave vectors belonging to \( D \) is finite. The so-defined \( \Omega_p \) obviously is a non-zero number as the minimum of a finite number of non-zero numbers.

Therefore, for a wide class of \( \omega \)'s it is possible to find the low boundary for the frequency imbalance \( \Omega \); in a finite spectral domain this boundary exists for arbitrary \( \omega \).

3. THE CLIPPING METHOD

For simplicity of the notation, we formulate our method here for the case of two-dimensional wave vectors \( k(m, n) \) and three-wave interactions only:

- (I) consider some specific PDE and write out the conditions of wave resonant interactions in the form (2) (the linear equation \( L(\psi) = 0 \) gives us the dispersion relation \( \omega = \omega(m, n) \) and the nonlinearity \( N(\psi) \) on the right-hand of (1) defines the number \( s \) of waves interacting resonantly);