
MAXWELL–LAGRANGE SYSTEM IN THE THEORY OF OPTICAL AND MAGNETIC RESONANCES

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The interaction of electromagnetic radiation with a medium that is approximated by an ensemble of two-level particles is treated on a quantum-mechanical basis. The system of equations of motion obtained for the phenomena of optical and magnetic resonances has a classical analog — the system of Lagrange-Poisson equations. In the approach, propagation effects are described by the nonlinear Maxwell-Lagrange system, which can be recast as coupled systems of linear equations by the inverse scattering method. The condition of compatibility of the solutions of the linear systems yields a hierarchical set of nonlinear evolution equations: the sine-Gordon equation, the nonlinear Schrödinger equation, the Korteweg-de Vries equation.

Introduction

The resonant interaction of the electromagnetic field with matter is usually studied in the semiclassical approximation, which is perfectly correct for strong fields. Nevertheless, quantization of the radiation field opens up new possibilities in the investigation of resonance, since, first, some phenomena (for example, superradiance), having a quantum nature, can be fully described only in the framework of quantum theory, and, second, the quantum-mechanical formulation of the problems of optical and magnetic resonances can, if suitably idealized, lead (as will be shown here) to exact solutions.

In addition, in a quantum-mechanical approach, the phenomena of not only optical but also magnetic resonance can be described in a unified manner on the basis of a system of differential equations that for all considered phenomena has the same form, namely, the form of the system of equations describing Lagrange-Poisson gyroscope type motion, and this, in its turn, leads to the discovery of exact analogs between not only the optical and magnetic effects but also between the quantum-mechanical and classical phenomena.

In contrast to the semiclassical approach, in which the characteristics of the field occur in the equation of the medium as parameter problems and vice versa, the quantum-mechanical equations form for the field and medium characteristics a closed and simultaneous system that possesses a sufficient number of common integrals of the motion to enable one, the initial conditions being specified, the corresponding exact solution for each phenomenon.

In the field of optical resonance, the inverse scattering method, by means of which one can successfully solve problems associated with the propagation of nonlinear waves, has been widely developed. By combining the Lagrange-Poisson equations with the Maxwell...
equations in the framework of the quantum approach one can use the inverse scattering method to investigate both optical and magnetic resonance.

1. Combined System of Maxwell–Lagrange Equations

The combined system of differential equations is found under the following assumptions. The time of resonant field-medium interaction and the evolution time of the system are much shorter than all the relaxation times. The medium can be approximated by an ensemble of two-level particles whose size is small compared with the radiation wavelength. The complete system of particles may be large. Only one radiation mode is chosen in the field. Under these conditions, the field-medium system is described by the Hamiltonian [1]

\[ \mathcal{H} = \hbar \omega a_k^+ a_k + \frac{1}{2} \hbar \omega \sum_{i=1}^{N} \alpha_{i,i} - \sum_{i=1}^{N} \{ (E_i, d_i) + (H_i, \mu_i) \}. \]  

(1.1)

Here \( a_k^+, (a_k) \) are the operators of creation (annihilation) of a photon of the field mode, \( N \) is the number of working particles, and \( E_i, H_i \) are the operators of the electromagnetic field intensities at the position of particle \( i \). For one mode of the field and fixed polarization, \( E_i, H_i \) are described as follows:

\[ E_i = \chi \left( \frac{2 \pi \hbar \omega}{V} \right)^{1/2} \{ a_k \exp[i(k, r_i)] - a_k^+ \exp[-i(k, r_i)] \}, \]

\[ H_i = i[k, \chi] \left( \frac{2 \pi \hbar c^2}{\omega V} \right)^{1/2} \{ a_k \exp[i(k, r_i)] - a_k^+ \exp[-i(k, r_i)] \}, \]

(1.2)

where \( r_i \) is the radius vector of the considered particle, and \( k \) is the wave vector. The operators of the dipole moment, \( d \), and magnetic moment, \( \mu \), can be written [1] as

\[ d = e_1 \sigma_1 + e_2 \sigma_2, \quad \mu = m_1 \sigma_1 + m_2 \sigma_2 + m_3 \sigma_3. \]  

(1.3)

Here, the vectors \( e_1 \) and \( m_1 \) can be expressed in terms of the dipole and magnetic moments of the transition between the states of the two-level particles, and \( m_3 \) can be expressed in terms of the magnetic moments of these states; \( \sigma_{i} \) are the Pauli operators.

The magnetic field which acts on the sample can always be represented in the form

\[ H = H_0 + H_\alpha \alpha, \]

(1.4)

where \( H_0 \) is a constant field, and \( H_\alpha \) is a vector operator perpendicular to \( \alpha \). It is also assumed that \( m_3 = \frac{\hbar \gamma}{2} \alpha \), where \( \gamma \) is the gyromagnetic ratio.

We define the operators

\[ f_{i,j} = \beta_j^* a_k^+ \exp[-i(k, r_i)] + \beta_j a_k \exp[i(k, r_i)], \]

(1.5)

Here

\[ \beta_j = i \left\{ \left( \chi, e_j \right) \left( \frac{2 \pi \hbar \omega}{V} \right)^{1/2} + \left( \left[ k, \chi \right], m_j \right) \left( \frac{2 \pi \hbar c^2}{\omega V} \right)^{1/2} \right\}, \quad j = 1, 2; \quad \Omega = \gamma H_0, \quad \xi_j = \{ f_{ij}, f_{ji}, f_{ij} \}. \]

In the notation (1.5), the Hamiltonian (1.1) can be rewritten as

\[ \mathcal{H} = \hbar \omega a_k^+ a_k - \sum_{i=1}^{N} (\sigma_i, f_i), \]

(1.6)

where \( \sigma_{i} = \{ \sigma_{i1}, \sigma_{i2}, \sigma_{i3} \} \). Expressing \( \sigma_{1,2} \) in terms of the operators \( \sigma_{\pm} \) and ignoring terms of the type \( \sigma_{\pm} a^\pm, \sigma_1 a \) (as is done in the rotating wave approximation [2]), we obtain instead of (1.6)

\[ \mathcal{H} = \hbar \omega a_k^+ a_k - \sum_{i=1}^{N} \sigma_{i, f_{ij}} - \sum_{i} \left\{ \frac{i}{2} \sigma_{\pm} \exp[i(k, r_i)] \beta_i a_k + \frac{i}{2} \sigma_{\pm} \exp[-i(k, r_i)] \beta_i^* a^+_k \right\}. \]

(1.7)