ON THE CLASSIFICATION OF MEROMORPHIC $c=24$ CONFORMAL FIELD THEORIES

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The simplest conformal field theories, from the point of view of the modular group or the fusion rules, are those with just one primary field with respect to some integer spin chiral algebra. It is elementary to show that unitary conformal field theories of this kind must have a central charge that is a multiple of 8. They transform according to a one-dimensional representation of the modular group with $S = 1$ and $T$ a cubic root of unity. Furthermore, if the central charge is a multiple of 24 the single character is modular invariant by itself, and can be written as a polynomial in the absolute modular invariant $j$,

$$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \ldots,$$

with a leading term $q^{-n}$ if the central charge $c = 24n$; here $q = e^{2\pi i \tau}$. Since the character $\mathcal{X}$ is modular invariant by itself one may consider, instead of the usual “diagonal” CFT with partition function $\mathcal{X}\mathcal{X}^*$, a purely chiral conformal field theory with partition function $\mathcal{X}$. Such a theory will be called a meromorphic conformal field theory, and denoted MCFT.

The classification of these theories is an essential part of the programme of classification of rational conformal field theories, initiated a few years ago. Indeed, one can argue that the entire RCFT classification problem can be embedded in that of classification of MCFT’s, provided that one can show that any RCFT has a complement. This is a RCFT with the same number of primary fields and complex conjugate $S$ and $T$ matrices. (A complement can easily be constructed for all WZW-models and for all coset theories without field identification fixed points.) Then any diagonal RCFT with a modular invariant $\sum_i \mathcal{X}_i(\mathcal{X}_i)^*$ can be mapped to a meromorphic one with partition function $\sum_i \mathcal{X}_i\mathcal{X}_i^C$, where ‘$C$’ denotes the complement.

In any case it is clear that the RCFT classification problem is not solved as long as we cannot even classify the theories with just one primary field. This is unfortunate, since for $c \geq 32$ the number of such theories grows so fast with the central charge that listing them is simply impossible. Indeed, for $c = 32$ the number of such theories is known to be larger than $8 \times 10^7$. The problem looks substantially easier for $c \leq 24$, and with some (unfounded) optimism one may hope that the information contained in the $c \geq 32$ theories will never really be needed in practice.

The fact that enumeration is impossible for $c \geq 32$ may dampen ones enthusiasm for attempting a enumeration for $c \leq 24$. Nevertheless, there are indications that the $c \leq 24$ theories (and in particular those with $c = 24$) are of some intrinsic interest. In physics, $c = 24$ is special because of the bosonic string, whose transverse dimension is 24; in mathematics the number 24 plays a special role in many contexts, such as the theory of sphere packings or the Monster group (the largest of the sporadic simple finite groups), for which a meromorphic $c = 24$ theory provides a “natural” $q$-graded representation, the “monster module” [1]. These rather vague motivations will probably turn out to be the most important ones for attempting to classify the meromorphic $c = 24$ CFT’s. A somewhat more practical motivation is that a listing of such theories will enable us to complete another classification problem, that of ten-dimensional heterotic strings. Yet another unsolved problem about which we will learn a few interesting new facts (without solving it, though) is that of the classification of Kac-Moody modular invariants.

A large class of MCFT’s can be constructed by taking $8n$ free bosons with momenta quantized on an even self-dual lattice. This gives 1, 2 and 24 [2, 3] distinct theories for $c = 8, 16$ and 24 respectively (and

more than $8 \times 10^7$ for $c = 32$). This class can be enlarged by a $\mathbb{Z}_2$ orbifold twist, using the symmetry that sends every boson $X$ to $-X$ [4, 5]. This gives back the same $E_{8,1}$ theory for $c = 8$, and maps the two $c = 16$ MCFT's ($E_{8,1})^2$ and $D_{16,1}(0) + [s]$) to one another (the argument denotes the conjugacy classes that appear). The result is more interesting when this twist is applied to the Leech lattice and the 23 Niemeier lattices: The former gives a new MCFT, the monster module, while from the latter one gets other Niemeier lattices in 9 cases, and new MCFT's in the 14 remaining cases [5]. Altogether this gives us thus 1, 2 and 39 MCFT's for $c = 8, 16$ and 24.

Clearly there are other orbifold twists one might consider, but it becomes rather difficult to prove the consistency of the resulting theories. More importantly, even an exhaustive classification of all orbifolds of known theories is not sufficient to show that the result is complete. The same is true for other kinds of constructions. For example, one could study all tensor products of Kac-Moody algebras with total central charge $8n$, and determine their meromorphic modular invariants. Even though this is a finite problem, there is no guarantee that the answer will be complete, since in general only part of the central charge will be saturated by (non-abelian) Kac-Moody algebras. As soon as one allows rational $U(1)$ factor the problem is not finite anymore, and it gets worse still if one adds factors without spin-1 currents (e.g. coset theories). In any case, it was already known for some time that the number of MCFT's with $c = 24$ is larger than the 39 mentioned so far: two additional candidates were presented in [6], one of which can certainly be constructed.

While explicit constructions approach the set of solutions from below it is possible in some cases to limit the set of solutions from above, i.e. to derive necessary rather than sufficient conditions for the existence of solutions. An example is the set of $c = 8$ and $c = 16$ solutions. Any such theory can be used to build a supersymmetric heterotic string theory in 10 dimensions. It can be shown in general ([7], see also [8]) that modular invariance of such a theory implies that all gauge and gravitational anomalies of the resulting field theory must factorize à la Green–Schwarz [9]. But all possibilities for such anomaly cancellations are known [9, 10], and this immediately reduces the $c = 8$ and $c = 16$ theories to $(E_{8,1})^2$ and $D_{16,1}$. There cannot exist more such theories, and since both can be constructed using self-dual lattices, there are no fewer either.

It turns out that a similar argument can be applied, with a considerably larger effort, to the $c = 24$ theories [23]. Beyond $c = 24$ the nature of the problem changes drastically, and these methods become useless, not just in practice but even in principle. The basic idea is to write down a character valued partition function for a given $c = 24$ theory analogous to similar functions introduced in [7] for the chiral sector of heterotic strings. This function generalizes the ordinary one-loop partition function

$$P(q) = \sum_{n=1}^{\infty} d_n q^n,$$

by replacing the multiplicities $d_n$ by Chern-characters of the representation at each level. Thus we get

$$P(q, F) = \sum_{n=1}^{\infty} \text{Tr} e^{F} q^n.$$

Here $F$ is some representation matrix of a semi-simple Lie-algebra, in the representation of the $n^{\text{th}}$ level.

To write down such a partition function we must have a Lie algebra that organizes the levels according to its representations. This happens if the theory has a set of spin-1 currents, which necessarily close into a Kac-Moody algebra, plus possibly some $U(1)$-currents [12]. Note that at this point we are certainly not assuming that these algebras saturate the central charge.

In general, the Kac-Moody algebra consists of several simple factors, and the partition function can be expressed in terms of the characters $\lambda_{\ell}^F$ of the $\ell^{\text{th}}$ factor and an unknown function without spin-1