A NONLINEAR PROBLEM CONCERNING THE COLLISION
OF TWO-DIMENSIONAL JETS OF AN IDEAL INCOMPRESSIBLE
FLUID WITH A FLOW DISCONTINUITY ON THE BOUNDARY
BETWEEN THE JETS

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Usually a curve of discontinuity in jet flows separates the moving from the quiescent fluid. However, there are cases in which the curves of discontinuity appear internal to the flow. In this paper we reduce a problem of this kind to a nonlinear system of singular integral equations, solvable by the method of a small parameter. Moreover each coefficient in the series involving the small parameter may be determined from a linear singular integral equation of the first kind.

Let a jet of fluid of density \( \rho_1 \) and speed \( v_1 \) flow onto the free surface \( FD \) from a channel of width \( \delta_1 \) (see Fig. 1), formed by the walls \( AC \) and \( AF \), parallel to the x axis. From a channel of width \( \delta_2 \), formed by the parallel walls \( BC \) and \( BG \), a jet of fluid of density \( \rho_2 \) flows onto the free surface \( GD \) with speed \( v_2 \). The walls \( AC \) and \( BC \) meet at angle \( \alpha \). The pressure on the free surfaces \( FD \) and \( GD \) is taken to be zero. Let

\[
\varepsilon = \frac{\gamma_1 \rho_1 v_1^2}{\gamma_2 \rho_2 v_2^2}
\]

where the numerator and denominator of the fraction are Bernoulli constants for the colliding jets. We may assume with no loss of generality that \( 0 \leq \varepsilon \leq 1 \). For \( \varepsilon = 1 \) the flow has, by Bernoulli's theorem, a discontinuity along the curve \( CD \) separating the jets and is describable by means of piecewise analytic functions, which in the sequel will be noted by "plus" and "minus" signs, respectively, in the flow regions \( ACDF \) and \( BCDB \). In accord with Bernoulli's theorem the point \( C \) is a critical point only for the jet flowing out of the channel with the walls \( BC \) and \( BG \). Consequently, the curve \( AC \) has a tangent whose slope varies continuously, and the curves \( BC \) and \( CD \) meet at point \( C \) at an angle equal to \( \pi - \alpha \). When \( \varepsilon = 1 \) and \( \rho_1 = \rho_2 \) the flow discontinuity along the curve \( CD \) is not present and the problem is solvable by the usual methods of the theory of jets of an ideal fluid [1]. This case was considered in [2-4], with an account of compressibility being included in the last of these papers. If \( \varepsilon = 1 \) and \( \rho_1 \neq \rho_2 \) then although the discontinuity along the curve \( CD \) is still present the flow may be described by means of analytic functions continuous on the curve \( CD \), as is done, for example, in the theory of the shaped charge [5].

If \( \varepsilon \to 0 \), it is then almost obvious that the curve \( CD \) tends toward the positive x semiaxis, and in the limiting case \( \varepsilon = 0 \) the flow regions \( ACDF \) and \( BCDB \) are bounded either by straight line segments or by free streamlines. Therefore when \( \varepsilon = 0 \) the flow may also be studied by the usual methods. This fact was used in [6] to solve this problem, to a first approximation, for a small value of \( \varepsilon \).

In this paper we reduce the problem to a nonlinear system of equations containing singular integrals, and we obtain a solution of this system when \( \varepsilon = 0 \). Assuming the solution in the neighborhood of the point \( \varepsilon = 0 \) to exist, and uniquely so, we obtain the asymptotic dependence on \( \varepsilon \) of the angle of inclination \( \varphi_D \) of the jet in the form of an expansion in powers of \( \varepsilon \).
1. We consider the flow at a given point of the region ACDFA. The corresponding domain of variation of the complex potential \( w = \phi + i \psi \) consists of a strip of width \( q_1 = v_A \delta_t \) (see Fig. 2), where \( q_1 \) is the outflow of fluid from the channel and \( v_A \) is the speed of the flow at an infinitely distant point of the channel. Let \( w(C) = q_1 \) and \( w(F) = \psi_F \), where \( \psi_F \) is a real number. The domain of variation of the function \( w \) may be mapped conformally onto the second quadrant of the auxiliary complex plane \( \tau = \xi + i \zeta \) with a correspondence of points in accord with Fig. 3. This mapping is unique, the quantities \( w \) and \( \tau \) being connected by the relation

\[
\tau = \frac{-\exp(\pi w/q_1) + 1}{\exp(\pi w/q_1) - \Psi_-} \quad \left( \Psi_- = \exp(\pi \psi_F/q_1) \right)
\]

(1.1)

where \( \Psi_- \) is a quantity depending on \( c \). It is readily seen that in this mapping the point \( A \) goes over into the point \( i \Psi_- - \frac{i}{2} \). Let the function

\[
\omega_\tau = \ln \left( \frac{dw_\tau}{v_\tau dz} \right) = \ln \frac{v_\tau}{v_1} - i \theta_\tau
\]

(1.2)

be analytic in the region ACDFA, where \( v_\tau \) and \( \phi_\tau \) are the modulus and argument of the vector velocity. The real and imaginary parts of this function vanish, respectively, on FD and FAC. The function \( \omega_\tau(\tau) \) is defined on the second quadrant and assumes real values on the positive part of the imaginary \( \zeta \) axis. In accord with the symmetry principle, therefore, it can be continued conformally onto the whole upper half-plane \( \tau \), so that the values of the function \( \omega_\tau(\tau) \) at points symmetric with respect to the imaginary axis will then be conjugate complex. Thus the real and imaginary parts of the function \( \omega_\tau \) will be, respectively, even and odd functions on the real \( \xi \) axis. Moreover, \( \text{Re} \ \omega_\tau = 0 \) on the intervals \((-\infty, -1]\) and \([1, \infty)\), corresponding to a free streamline.

Using Schwarz' integral for the halfplane we can represent the function \( \omega_\tau(\tau) \) in the form [5]

\[
\omega_\tau(\tau) = \frac{1}{\pi i} \int_{-1}^{1} \ln v_\tau \frac{d\xi}{\xi - \tau} + Ci
\]

(1.3)

It is convenient from now on to use a dimensionless quantity in place of the pressure \( p \), which we define by

\[
\lambda = \frac{2p}{\rho v_t^2}
\]

For motion along the curve CD the quantity \( \lambda \) varies from 1 to 0. Let \( \xi_- (-1 \leq \xi_- \leq 1) \) be a point of the real axis in the \( \tau \) plane corresponding to a point of the flow discontinuity curve in the physical \( z \) plane. The quantities \( \xi, \lambda, \) and \( \xi_- \) are then connected by the relations

\[
f_-(\xi, \lambda, \xi_-) = 0, \quad f_-(\xi, 1, 0) = 0, \quad f_-(\xi, 0, \pm 1) = 0
\]

(1.4)

The last two of these relations result from the correspondence of points under the conformal mapping. Assume now that the relations (1.4) define continuous functions \( \xi_- = \xi_-(\xi, \lambda) \) and \( \lambda = \lambda(\xi, \xi_-) \), the latter of these being an even function of the argument \( \xi_- \).

If we express \( v_\tau/v_1 \) in terms of \( \lambda \) with the aid of Bernoulli's theorem and then substitute the result into relation (1.3), we find

\[
\omega_\tau(\tau) = \frac{1}{2\pi i} \int_{-1}^{1} \ln(1 - \xi \lambda) \frac{d\xi_-}{\xi_- - \tau} + Ci
\]

(1.5)

\[
\omega_\tau(\tau) = \frac{1}{2\pi i} \int_{-1}^{1} \ln(1 - \xi \lambda) \frac{d\xi_-}{\xi_- - \xi_0} + Ci = \frac{\xi}{i} \int_{0}^{1} \ln(1 - \xi \lambda) \frac{d\xi_-}{\xi_-^2 + \xi^2} + Ci
\]

(1.6)

\[
\frac{v_\lambda}{v_1} = \exp \frac{1}{2\pi i} \int_{-1}^{1} \ln(1 - \xi \lambda) \frac{d\xi_-}{\xi_- - \xi \Psi_- \xi_0} = \exp \frac{1}{\pi} \int_{0}^{1} \ln(1 - \xi \lambda) \frac{d\xi_-}{\xi_-^{\Psi_- \xi_0} + 1}
\]

(1.7)

In the last integral in Eq. (1.6) we have used the fact that \( \lambda \) is an even function. Since the point \( i \xi \) corresponds to the wall CAF, parallel to the \( x \) axis, it follows from the expression (1.6) that \( C = 0 \).