This paper is devoted to a study of the vortex-free (irrotational) motion of an ideal incompressible liquid during the vertical immersion of a cylindrical solid. In contrast to problems of impact [1] and the entry of a solid into water [2], the case here treated deals with continuous immersion involving a change in the shape of the free surface but with a constant width of the wetted surface of the solid. The coefficients of the time-dependent power series for the velocity potential, the equation of the free surface, and the pressure on the solid are determined, allowing for all the terms in the Cauchy-Lagrange equation. The results of calculations relating to the immersion of a bottom with an elliptical shape of the submerged part are presented.

1. We shall consider a cylindrical solid floating on the surface of a liquid at rest, its submerged part being bounded by a smooth and symmetrical curve ABC, \( y = y_1(x) \); at the points A and C the solid has vertical walls. Immersion takes place at a rate \( v(t) \) for \( t \rightarrow 0 \).

We shall consider an ideal and incompressible liquid. The velocity potential \( \varphi(x, y, t) \) satisfies the equation

\[
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0.
\]

(1.1)

Assuming that the flow around the solid takes place without detachment, we obtain the following relationship on the surface of the solid (\( n \) is the external normal to the curve ABC):

\[
\frac{\partial \varphi}{\partial n} = v_n(t) \quad \left( y = y_1 - \int_0^t v(\tau) d\tau \right).
\]

(1.2)

On the free surface of the liquid \( y = \eta(x, t) \) the following dynamic and kinematic conditions are satisfied (\( g \) is the gravitational acceleration):

\[
\frac{\partial \eta}{\partial t} + g + \frac{1}{2} \left( \frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial \varphi}{\partial y} \right)^2 = 0
\]

\[
\frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial x} = \frac{\partial \eta}{\partial y}.
\]

(1.3)

The system (1.3) at \( t = 0 \) and \( \sqrt{x^2 + y^2} \rightarrow \infty \) has the initial and boundary conditions \( \eta = \varphi = 0 \).

2. We shall seek a solution by the small-parameter method, using the time \( t \) as parameter.

Let us consider the following series:

\[
\varphi = \sum_{k=0}^{\infty} \mathcal{Q}_k(x, y), \quad \eta = \sum_{i=1}^{\infty} \mathcal{I}_i(x), \quad v = \sum_{m=1}^{\infty} t^m \frac{d^n v(0)}{dt^n}.
\]

(2.1)

Here \( \mathcal{Q}_k \) and \( \mathcal{I}_i \) are unknown functions. It follows from (1.1) and (2.1) that \( \mathcal{Q}_k \) are harmonic functions. Let us expand \( \mathcal{Q}_k \) in a Taylor series in the neighborhood of the curve \( L \), constituting the upper boundary
to the region D occupied by the unperturbed liquid at the initial instant of time. The following equations are then satisfied:

\[
\frac{\partial q}{\partial t} = \sum_{m=0}^\infty \sum_{n=-m}^m \frac{(y-y_0)^n}{s!} \frac{\partial^{m+n} q_n(y_2)}{\partial y^n \partial y^m},
\]

\[
\frac{\partial q}{\partial q} = \sum_{m=0}^\infty \sum_{n=-m}^m \frac{(y-y_0)^n}{s!} \frac{\partial^{m+n} q_n(y_2)}{\partial q \partial y^m}.
\]

Here q is one of the vectors x, y, or n; y = y_0(x) is the equation of curve L. Since on the curve ABC \(y_0 = y_1(x)\), while on the free surface \(y_0 = 0\), we have

\[
y - y_0 = \sum_{m=0}^\infty \frac{d^{m+1} v(0)}{dt^{m+1}}, \quad |x| \leq l
\]

\[
y - y_0 = \sum_{m=1}^\infty \frac{d^m v(0)}{dt^m}, \quad |x| > l \quad (l = AC/2).
\]

The boundary conditions for \(\varphi_k\) on the boundary L and the values of \(r_1\) are determined by substituting the series (2.2) and (2.3) into Eqs. (1.2) and (1.3) and equating the coefficients of equal powers of t.

The corresponding relationships for several of the first values of k and m have the following form:

For \(|x| \leq l\)

\[
\frac{\partial q_0}{\partial n} = v_0(0), \quad \frac{\partial q_1}{\partial n} = v_0(0) \frac{\partial^2 q_1}{\partial y \partial n} + \frac{dv_0(0)}{dt},
\]

\[
\frac{\partial q^2}{\partial n} = \frac{1}{2} \frac{d^2 v_0(0)}{dt^2}, \quad \frac{\partial q_3}{\partial n} = \frac{1}{2} \frac{d^2 v_0(0)}{dt^2} \frac{\partial^2 q_3}{\partial y^2} + \frac{d^2 v_0(0)}{dt^2} l \frac{\partial^2 q_3}{\partial y \partial x} + \frac{d^2 v_0(0)}{dt^2} l \frac{\partial^2 q_3}{\partial x^2}.
\]

For \(|x| > l\)

\[
q_2 = 0, \quad q_3 = -\frac{1}{2} \left( \frac{\partial q_3}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial q_3}{\partial y} \right)^2, \quad q_2 = 0
\]

\[
r_1 = \frac{\partial q_3}{\partial y}, \quad r_2 = \frac{1}{2} \left( \frac{\partial q_3}{\partial y} + \frac{\partial q_3}{\partial x} \right), \quad r_3 = \frac{1}{3} \frac{\partial q_3}{\partial y}.
\]

The relationships for \(\varphi_2, \varphi_3,\) and \(r_3\) in (2.4) and (2.5) are calculated for \(v(0) = 0\).

The boundary conditions (2.4) enable us to follow the relationship between the phenomena of impact and immersion. According to [1], \(\varphi_0\) is the velocity potential for the impact of a solid with velocity \(v(0)\).

In this case the continuity of the functions \(\varphi\) and \(v\) at \(t = 0\) is infringed; the series (2.1) and the following expansions will be valid for \(t > 0\). It also follows from (2.4) that if \(\frac{d^m v(0)}{dt^m} = 0\) for all \(m < \lambda\), then \(\varphi_1\) will be equal to the velocity potential for the impact of a solid at a velocity numerically equal to \(\lambda \frac{dv(0)}{dt}\), and \(\varphi_k = 0\) for all \(k \leq \lambda\).

In order to find \(\varphi_k\) we transform the region D conformally into the lower half-plane \(\xi = u + iv\) by means of the function \(z = z(\xi)\) \((z = x + iy)\), so converting the curve ABC into the segment \((-1, 1)\) of the \(u\) axis. The value of the normal derivative to the curve ABC may be calculated from

\[
\frac{\partial q_k}{\partial \xi} = \frac{d z}{d u} \left| \frac{d z}{d \xi} \right|.
\]

The Keldysh-Sedov formula [3] enables us to calculate the functions \(\frac{\partial \Phi_k}{\partial \xi}\) \((\text{Re} \Phi_k = \varphi_k)\) successively for \(k = 0, 1, 2, \ldots\) in a class of functions having an integrable singularity at \(|u| = 1\). Since \(\frac{\partial \varphi_k}{\partial u}\) is an odd function and \(\frac{\partial \varphi_k}{\partial v}\) an even one, this equation transforms into

\[
\frac{\partial \Phi_k}{\partial \xi} = \frac{2 \xi}{\pi \xi^2 - 1} \left\{ \int_1^{\infty} \frac{\tau^{\xi^2 - 1} - \tau^{\xi^2 - 2}}{\tau^{\xi^2} - \xi^2} \frac{d \tau}{\tau} + \int_0^1 \frac{1}{\tau^{\xi^2 - 1}} \frac{1}{\tau^{\xi^2 - 2}} \frac{d \tau}{\tau} \right\}.
\]