Many studies, both theoretical and experimental, have been dedicated to the stability of flow in a circular tube (see, for example, review [1]). In every case mathematical investigation has not succeeded in obtaining an expression for hydrodynamic instability of such a flow for disturbances of sufficiently low amplitude. (An exception is [2].) Experiment also indicates the stability of such a flow [3], with a laminar mode being extended to Reynolds numbers of the order of tens of thousands. These facts are the basis for the assumption that the flow of a viscous incompressible liquid in a circular tube is stable for small perturbations. However, there is no analytical or even numerical proof of this hypothesis. Moreover, some studies, for example [2], indicate the instability of such a flow in relation to three-dimensional nonaxisymmetric perturbations. The analysis of hydrodynamic stability with respect to three-dimensional disturbances of flow within a circular tube conducted in this study showed the stability of the flow over a wide range of wave numbers and Reynolds numbers.

1. Formulation of the Problem

Let there be applied to the fundamental flow with velocity vector \( \mathbf{V}_0 = \{0, 0, u(r)\} \) and pressure \( P_0 \) small perturbations of the type

\[
\mathbf{v} = w(r) G, \quad p = -\frac{i}{\alpha R} q(r) G, \quad G = \exp[ia(z - ct) + im\theta]
\]

Corresponding equations for these disturbances in a cylindrical coordinate system \((r, \theta, z)\) were obtained in [4]:

\[
\begin{aligned}
\left(\frac{1}{r^2} - \frac{2im}{r^2} w_1 + \frac{i}{\alpha} q \right)' &= 0 \\
\left(\frac{1}{r^2} w_2 \right)' - bw_1 - \frac{2im}{r^2} w_1 + \frac{m}{\alpha r} q &= 0 \\
\left(\frac{1}{r^2} w_3 \right)' - bw_1 - R \omega_1 - q &= 0 \\
(w_1)' + imw_1 + iarw_3 &= 0 \\
b &= iaR(u - c) + \frac{m^2}{r^2} + r^2
\end{aligned}
\]

(1.1)

Here \( w = \{w_1, w_2, w_3\} \), wave numbers \( a \in (0, \infty) \), \( m = 0, 1, \ldots \), \( c = X + iY \) is the sought for eigenvalue (at \( Y < 0 \) the disturbance decays with time); \( u \) is the undisturbed velocity; \( h \) is the tube radius; \( U \) is the mean velocity; and the prime indicates differentiation with respect to \( r \).

Below, instead of \( w_i \) we will utilize new functions, for which the following system is obtained from Eq. (1.1):

\[
\Psi = \frac{i}{\alpha} r \omega, \quad \Phi = r \omega, \quad f = r \omega', \quad F = \frac{\Phi'}{r}, \quad W = \omega, \tag{1.2}
\]

\[
f' = rbW - iaRu'\Psi + rq, \quad F' = \frac{1}{r} \left( b\Phi - \frac{2am}{r^2} \Psi + \frac{m}{\alpha} q \right)
\]

\[
q' = \frac{\alpha^2}{r} \left( f - b\Psi + \frac{m}{\alpha} \right), \quad \Psi' = rW + \frac{m}{\alpha r} \Phi, \quad \Phi' = rF \tag{1.3}
\]

The adherence condition on the tube wall is written as

\[
W = \Psi = \Phi = 0 \quad (r = 1) \tag{1.4}
\]

The three remaining conditions are obtained from the requirement that the velocity be finite at \( r = 0 \). According to Eq. (1.2) at \( r = 0 \) this leads to the equality

\[
\Psi = \Phi = f = 0
\]

For numerical solutions it is necessary to be able to calculate the right-hand portions of Eq. (1.3) at \( r = 0 \). Significant difficulties in elimination of singularities arise here.

If an expansion of the solution in a power series in the vicinity of \( r = 0 \) is used, then for each \( m \) individual expansion coefficients are obtained, which have a quite cumbersome form. Moreover, at high values of \( \alpha R \) a large number of series terms must be considered. However, the basic difficulty consists of the fact that in finding the fundamental solutions of system (1.3) the values of \( W, F, q \) at \( r = 0 \) cannot be set arbitrarily. This is due to the fact that the function \( W(r) \) near \( r = 0 \) has a form \( W \sim r^m \) and for \( m > 0 \) it is necessary to set not \( W(0) \), but the corresponding derivative.

In our case these difficulties will be surmounted by setting the boundary conditions not at \( r = 0 \), but at \( r = r_0 \ll 1 \)

\[
\Psi = \Phi = f = 0, \quad W = C_1, \quad q = C_2, \quad F = C_3 \quad (r = r_0) \tag{1.5}
\]

These conditions ensure finiteness of the solution and admit a smooth transition to the limiting case of an "empty" tube at \( r_0 \to 0 \), in contrast to the variant where the physical adhesion characteristics in the form of Eq. (1.4) are applied to the tube wall.

Thus the problem is solved by the eigenvalues of Eqs. (1.3-1.5). To calculate these values a method described in [4] is used.

In applying the method to an electronic computer control variants were developed and results of the calculation compared with results obtained earlier in [1].

For a wide range of parameter values eigenvalue calculations were conducted for three problems:

I) Eqs. (1.3)-(1.5); II) Eqs. (1.3), (1.4) and at \( r = 0 \) the boundary conditions obtained by an expansion in power series in \( r \) for the vicinity of \( r = 0 \); III) Eq. (1.3), and at \( r = 0, r = 1 \) the adhesion conditions of Eq. (1.4).

In problem II the undisturbed velocity profile is described by the function \( u = 2(1 - r^2) \); in problems I and III \( u = Ar^2 + B \ln r + C \). The constants \( A, B, \) and \( C \) were obtained from the conditions

\[
u(1 + \xi) = 0.2 \int_0^{r_0} ru dr = 1 + 2\xi \quad \text{(problem I)}
\]

\[
u(\xi) = 0 \quad \text{(problem III)}
\]

and the conditions \( u'(\xi) = 0 \) for problem I or \( u(\xi) = 0 \) for problem III.

Here the parameter \( \xi \) is the ratio of the internal cylinder radius to the distance between cylinders. For \( \xi \to 0 \) the velocity profile in all cases tends to parabolic, but discontinuously in the case of problem III.

Calculations showed that eigenvalues of problems I and II practically coincided and differed no more than 4% from the corresponding values of problem III.