Chapter 1

Introduction

1.1 General introduction

Einstein's theory of general relativity provides the geometrical setting for the gravitational force. At a very large scale, it describes the behaviour of stars, galaxies and ultimately the universe itself. This theory has been 'experimentally' verified in many ways, ranging from the classic motion of the perihelion of Mercury to, more recently, the strong gravitational effects of orbital decay in a binary system of neutron stars \(^{1)}\) [36]. This orbital decay is the best evidence to date for the existence of gravitational waves. General relativity has also provided us with the very successful Big Bang model for the evolution of the very early universe. The recent remarkable experimental verification [130] that the remnant cosmic background radiation has an essential anisotropy (a temperature variation \(\Delta T/T\) of order \(10^{-6}\)), has finally uncovered the density fluctuations in the very early universe which have been (and still are) the seeds for the formation of the galactic structures we observe at present.

In contrast to this enormous success, problems with general relativity arise if one wants to describe physical processes at very small distance scales. Indeed, one of the outstanding problems in theoretical physics is to reconcile the postulates of quantum mechanics with the theory of the gravitational force as given by Einstein. The most advanced methods of (perturbative) quantum field theory lead, when applied to the theory of general relativity, to incurable divergencies already at low orders in Newton's constant \(G_N\).

The basic problem derives from the fact that this coupling constant is not dimensionless \((G_N = 1/M_P^2)\), where \(M_P \sim 10^{19}\) GeV is the Planck mass, in units with \(\hbar = c = 1\). An alternative way to see that the intrinsic mass scale present in general relativity causes a problem of interpretation is to consider the quantum fluctuations at very small scales. Fluctuations of sizes smaller than the Planck length \(L_P = 1/M_P\) have energies \(E\) larger than \(M_P\). The Schwarzschild radius of such fluctuations is \(R_S = G_N E \geq L_P\), i.e. the quantum fluctuations of spacetime at a scale smaller than the Planck length become tiny black holes! This is not just a collapse of spacetime but rather of the notion of spacetime itself, posing a serious problem of interpretation.

The theoretical catastrophes indicated above are often taken to be an indication that we have to replace general relativity by a different, more fundamental theory if we want to understand quantum gravity. This would mean that general relativity is to be interpreted as an effective low energy theory. This viewpoint then expresses the quest for a new dynamics underlying the geometric principles of Einsteins theory!

Some ten years ago a new paradigm called superstrings [77] emerged at the horizon0f this conceptual black hole. Though maybe overadvertised by its proponents and overcriticized by its opponents, it still

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\(^{1)}\)J.H. Taylor has been awarded the 1992 Wolf Prize in Physics for this discovery.

appears to be a most promising candidate for a theory of quantum gravity today.

The revolutionary aspect of string theory resides in the basic assumption that the most fundamental entities in nature are not particles but one-dimensional extended objects called strings, which have a size of the order of the Planck length. In this theory one envisages that the notion of spacetime at the Planck length loses its meaning and gets replaced by some (nonperturbative) string dynamics.

In a quantum description of strings as advocated by Polyakov [119] one is forced to define a suitably weighted sum over the random surfaces corresponding to the worldsheet swept out by the string in spacetime. This can be done by introducing a metric on the worldsheet, which allows for a classically reparametrization- and Weyl invariant formulation. In this rather geometric approach the interactions between strings, i.e. the joining and splitting of them, are automatically taken care of by also summing over different worldsheet topologies, i.e. surfaces with handles.

As a first step in this programme one may consider the problem of string propagation in a fixed background spacetime, in such a way that there is no anomaly and the metric fluctuations of the worldsheet decouple (the so-called critical string). Clearly this implies a perturbative expansion around a particular classical groundstate of the theory. This approach has culminated in the development of (rational) conformal field theory on Riemann surfaces, which is governed by the Virasoro algebra and its extensions. Ultimately, this formulation produces a perturbation series in the number of string loops (i.e. the number of handles) which may be finite term by term but which is not Borel summable [79].

An alternative scheme to calculate the (noncritical) string partition function starts out from discretisation of the worldsheet by equilateral triangles. The resulting partition sum can be translated into a matrix integral. This matrix model exhibits multi-critical behaviour, which allows one to define a continuum limit in any of its critical points. The partition functions of the simplest matrix model obtained this way reproduce the perturbative (genus) expansion of some minimal (c < 1) conformal models coupled to two-dimensional quantum gravity (i.e. living on a fluctuating random surface). In other words, these models (can unfortunately only) describe string propagation in less than one dimension.

However, the crucial new result is that using powerful methods to analyse the matrix integrals, one can derive an exact differential equation in terms of the inverse string coupling constant (the string equation) for the full partition function, providing a starting point for a nonperturbative approach. In addition one may derive similar equations for the correlation functions of the theory. Remarkably enough, this infinite set of equations forms a so-called hierarchy of nonlinear evolution equations, which for the simplest model is nothing but the Korteweg-de Vries (KdV) hierarchy [48]. This deep result can be recast in a somewhat different form, as an infinite set of constraints which the partition function has to obey, corresponding to Ward identities which may be derived from symmetries underlying the original matrix integral. These constraints satisfy an algebra which is (part of) the Virasoro algebra or one of its extensions.

What the lightning review just given attempts to emphasize is, that depending on the way one looks at the problem of quantum strings (i.e. of quantum gravity), one gets involved in branches of physics which appear to be rather disconnected at first sight. On the one hand, this is conformal field theory which among other things gives a powerful description of two-dimensional critical phenomena, relating the universal critical exponents to the representation theory of the underlying conformal algebra. On the other hand, this is the theory of integrable nonlinear evolution equations, describing soliton propagation and scattering in such widely different situations as shallow canals and optical fibers.

One of the reasons that this is possible is the fact that these phenomena share an underlying algebraic structure. There were at first the infinite algebras of the Lie-type such as the Virasoro algebra, the affine Kac-Moody algebras, and their super counterparts. Whereas more recently the nonlinear extensions of the Virasoro algebra, the so-called \( \mathcal{W} \) algebras receive much attention. As will become clear it is the study of these algebras and their applications indicated above which is the heart of this review. We have thought it useful to start this first chapter by separately introducing the topics mentioned above, i.e. conformal field theory, integrable hierarchies and matrix models. In the later chapters we will then reveal their interwoveness.

The paper is organized as follows. First, we give a (short) introduction to two-dimensional conformal field theory, in particular the notion and relevance of extended conformal algebras or \( \mathcal{W} \)-algebras (sec-