ON OBSERVABLE ALGEBRAS OF A CLASS OF ASSOCIATIVE MECHANICAL SYSTEMS

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A classification is made of the associative algebras of functions invariant with respect to the adjoint action of the group of affine transformations of the line, these functions containing an everywhere dense subalgebra of polynomials. The applications of the results to the problem of describing possible Hamiltonian mechanical systems are discussed.

1. It is usually assumed (see, for example, [1], p. 387 of the Russian translation) that the observables of a physical theory are the elements of the universal covering (or its factor algebra) of some Lie algebra. From this point of view, nonrelativistic quantum mechanics has observable algebra generated by the Weyl algebra, the factor algebra of the universal covering $U(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of the Heisenberg group (see [2], §4.6). The universal covering $U(\mathfrak{g})$ is invariant with respect to the adjoint representation of the Lie group G, for which $\mathfrak{g}$ is the Lie algebra. In this example of nonrelativistic quantum mechanics, the adjoint representation is specified by the operators of parallel transport of the phase space of the system, and the observable algebra (which is equivalent to the algebra of pseudodifferential operators), which contains the Weyl algebra as an everywhere dense subalgebra, is realized in the form of a nonlocal algebra of functions on the phase space, this algebra being invariant with respect to the group of shifts with multiplication operation given by a bipseudodifferential operator. The presence in the observable algebra of invariance with respect to the group of shifts made it possible in [3-8] to classify physical theories that are nearly canonical. It is natural to carry out an analogous program by taking instead of the Heisenberg group and its adjoint representation some other Lie group. The main difficulty then encountered is that the functional equations constituting the basis of the approach developed in [3-8] become operator equations. In the first part of the work, we give a classification of associative algebras of functions that are invariant with respect to the adjoint representation of one of the subgroups of the Lorentz group—the group of affine transformations of the line—and contain an everywhere dense subset of polynomials. This is done on the basis of the obtained general solution of the corresponding operator functional equation. The second part of the paper is devoted to application of the obtained classification to the problem of describing the possible associative Hamiltonian mechanical systems [9]. The paper is a natural continuation of [3-8]. Since [8] contains a detailed review of the questions considered here, we turn to the direct exposition of the results.

2. The group $G$ of affine transformations of the line, $g(a, b): x \mapsto ax+b$ ($x \in \mathbb{R}$, $b \in \mathbb{R}_\ast$, $a \in \mathbb{R} \backslash \{0\}$), is the simplest two-parameter non-Abelian Lie group and has a faithful representation:

$$g(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}.$$  

We denote by $x$ and $p$ the generators of the Lie algebra $\mathfrak{g}$ of the group $G$; they satisfy the commutation relation

$$[p, x] = x. \tag{1}$$

We shall denote the elements of the universal covering $U(\mathfrak{g})$ (the fundamental field is the complex-number field $\mathbb{C}$) by upper case Latin letters, i.e., $U(\mathfrak{g}) = \{A(x, p)\}$. In writing down the elements $A \in U(\mathfrak{g})$, we introduce the following ordering: In each of the homogeneous components of $A(x, p)$, we take in the first position the degree of the generator $x$ (analog of qp quantization for expressing (pseudo)differential operators). The operation of multiplication $\pi_0$ in $U(\mathfrak{g})$ can be represented in the form of a bipseudodifferential operator. We begin by introducing some notation.

Let $\omega: U(\mathfrak{g}) \times U(\mathfrak{g}) \to U(\mathfrak{g})$ be the binary operation of the form

$$\omega(AB)(x, p) = \omega(1, 2) A(x_1, p_1) B(x_2, p_2) = A(x_1, p_1) B(x_2, p_2) |_{x_1 = x, p_1 = p, x_2 = p, p_2 = p}, \tag{2}$$

where $A(x, p)B(x, p)$ denotes the product of the elements $A$ and $B$ taken in $S(\mathfrak{g})$, a symmetric algebra of the linear space $\mathfrak{g}$ (i.e., the ordinary associative–commutative product of polynomials). In other words, $\omega AB = AB$, $A, B \in S(\mathfrak{g})$. By $S(\mathfrak{g})$, we shall also denote the linear space of the algebra $U(\mathfrak{g})$.

In $S(\mathfrak{g})$, the ordinary operations of differentiation $\partial_x = \partial/\partial x$, $\partial_p = \partial/\partial p$ are defined, and we have the properties

\[
\partial_x \omega A(x, p)B(x, p) = \omega (1, 2) (\partial_x + \partial_p) A(x, p_1)B(x, p_2),
\]
\[
\partial_p \omega A(x, p)B(x, p) = \omega (1, 2) (\partial_p + \partial_p) A(x, p_1)B(x, p_2).
\]

With respect to the multiplication $\pi_\theta$ in $U(\mathfrak{g})$, the unary operators $\partial_x$, $\partial_p$ are not operations of differentiation. By means of the operation $\omega$, the bipseudodifferential operator $F$ on $S(\mathfrak{g})$ can be written in the form

\[
(FAB)(x, p) = \omega (1, 2) F(x, p_1, \partial_x, \partial_p, \partial_x, \partial_p) A(x, p_1)B(x, p_2).
\]

where $F(x, p, \partial_x, \partial_p, \partial_x, \partial_p)$ is a formal series in the variables $\partial_x, \partial_p, \partial_x, \partial_p$ with coefficients in $\mathbb{C} [x, p]$ and its action as an operator $S(\mathfrak{g}) \times S(\mathfrak{g}) \rightarrow S(\mathfrak{g}) \times S(\mathfrak{g})$ is defined by the following convention: On $A(x_1, p_1)B(x_2, p_2)$, there first act the differentiation operators $\partial_x, \partial_p, \partial_x, \partial_p$. In what follows, we shall call $F(x, p, \partial_x, \partial_p, \partial_x, \partial_p)$ the kernel of the bipseudodifferential operator $F$ (of the binary operation defined by it). Using the relation (1), we can readily show that the following lemma holds.

**Lemma 1.** The operator of multiplication $\pi_\theta$ in $U(\mathfrak{g})$ can be represented as follows:

\[
\pi_\theta AB(x, p) = \omega (1, 2) \exp\{x_1 \partial_x + [x_2, \theta] \} A(x, p_1)B(x_2, p_2).
\]

The adjoint representation of the group $G$ in $\mathfrak{g}$,

\[
(x, p) \rightarrow (ax, p-bx),
\]

can be uniquely extended to a representation in $U(\mathfrak{g})$.

**Lemma 2.** The adjoint action of the group $G$ in $U(\mathfrak{g})$ is defined by operators $T_{\xi x, \xi p} \in G$, of the form

\[
T_{\xi x, \xi p} A(x, p) = \exp\{-b \partial_x \theta \} A(ax, p-bx).
\]

The kernels of the binary operations on $S(\mathfrak{g})$, expressed in the form (4), being operators on the tensor product $S(\mathfrak{g}) \otimes S(\mathfrak{g})$, naturally generate an algebra with respect to composition, which in what follows we shall denote by the asterisk $*$. The same is obviously true for the kernels of the unary operations, the set of which is embedded in the set of the kernels of the binary operations. Note that the convention in (4) specifies "qp ordering" in expressions of the noncommuting operators $x, \partial_x, \partial_p$, or "qp quantization" [10]. In some cases, the operation $*$ can also be represented in the form of a bipseudodifferential operator:

\[
F_1(x, p, \partial_x, \partial_p, \partial_x, \partial_p) \ast F_2(x, p, \partial_x, \partial_p, \partial_x, \partial_p) = \exp\{D_x \partial_x + D_p \partial_p\} F_1(x, p, \partial_x, \partial_p, \partial_x, \partial_p) F_2(x, p, \partial_x, \partial_p, \partial_x, \partial_p),
\]

where $D_x = \partial/\partial \partial_x$, $D_p = \partial/\partial \partial_p$ act on the first factor $F_1$, and $\partial_x^2, \partial_p^2$ act on the second $F_2$. Equation (8) makes it possible to carry out directly the $*$ multiplication when, for example, $F_1$ and $F_2$ are polynomials with respect to their arguments or exponentials of polynlinear forms in the variables. In the present paper, we investigate associative binary operations of the type (4) invariant with respect to the adjoint action of the group $G$ (7). However, before we study the class of binary $G$-invariant operations of the form (4), we consider unary $G$-invariant operations. We first need

**Lemma 3.** The algebras of the kernels of the binary operations $\{F(x, p, \partial_x, \partial_p, \partial_x, \partial_p), \ast\}$ and the unary operations $\{\Omega(x, p, \partial_x, \partial_x), \ast\}$ (as noted, $\{\Omega, \ast\}$ is $\{F, \ast\}$) of being defined by the equation $\omegaF = \omega F$ and $\Omega(x, p, \partial_x, \partial_x) = \Omega(x, p, \partial_x, \partial_x)$ in $\{F, \ast\}$ are equivalent to the algebras $\{\tilde{F}(x, p, \partial_x, \partial_p, \partial_x, \partial_p), \ast\}$ and $\{\tilde{\Omega}(x, p, \partial_x, \partial_x), \ast\}$, respectively, in the following sense: There exists a bijective operator $W: \{F, \ast\} \rightarrow \{F, \ast\}$ such that

\[
W(F(x, p, \partial_x, \partial_p, \partial_x, \partial_p)) = WF(x, p, \partial_x, \partial_p, \partial_x, \partial_p),
\]
\[
W(F_1 \ast F_2) = W(F_1) \ast W(F_2),
\]
\[
\frac{\partial W}{\partial p}(x, p, \partial_x, \partial_p, \partial_x, \partial_p) = W(F(x, p+1, \partial_x, \partial_p, \partial_x, \partial_p) - F(x, p, \partial_x, \partial_p, \partial_x, \partial_p)).
\]

The explicit form of the operator $W$ is