in the kinetic equations it is necessary to omit the terms containing the spatial derivatives of the macroscopic variables.

LITERATURE CITED


QUANTUM NONLINEAR SCHRODINGER EQUATION ON A LATTICE

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A local Hamiltonian is constructed for the nonlinear Schrödinger equation on a lattice in both the classical and the quantum variants. This Hamiltonian is an explicit elementary function of the local Bose fields. The lattice model possesses the same structure of the action-angle variables as the continuous model.

1. Introduction

In the continuous case, the quantum nonlinear Schrödinger equation is equivalent to a one-dimensional Bose gas with δ-function potential [1]. The model was studied from the point of view of the quantum inverse scattering method in [2, 3]. Trace identities were proved in [4].

The first variant of the lattice nonlinear Schrödinger equation was constructed in [5], the corresponding quantum model in [6]. It was found, however, that the continuous and lattice models possess different R matrices [6]. In [7], the nonlinear Schrödinger equation on a lattice with the same R matrix as in the continuous case was constructed. The absence of change in the R matrix when the model is transferred to a lattice means that the structure of the action–angle variables is preserved [8], this being particularly important for quantization. The Hamiltonian of the model proposed in [7] is local only in the classical case, a nonlocal expression being obtained for the Hamiltonian in the quantum case. Soon, a local Hamiltonian for both the classical and the quantum lattice model was successfully constructed [9]. The construction of the local Hamiltonian was based on the following observation. When the quantum determinant vanishes, the L operator is transformed into a one-dimensional projection operator. In [10], a different approach was used to construct a local Hamiltonian for the model of [7]. An important part in this approach is played by the R matrix, which intertwines the L operator of the model with respect to the quantum space. By means of this R matrix, one constructs a fundamental spin model, and this generates the local Hamiltonian.

In the present paper, we develop the approach of [9]. We have succeeded in significantly simplifying not only the explicit expression for the local quantum Hamiltonian in terms of the original Bose fields but also the dependence of the single-particle energy on the spectral parameter. In the quantum case, the lattice model can be explicitly solved by means of the algebraic Bethe ansatz [11]. The plan of our paper is as follows. In Sec.2, we investigate the classical model. In Sec.3, we construct the local quantum Hamiltonian. In Sec.4, the model is solved by means of the algebraic Bethe ansatz.
2. Classical Model

The Hamiltonian of the continuous nonlinear Schrödinger equation has the form

\[ H = \int dx (\partial_t \psi^* \partial_x \psi + c \psi^* \psi) \]  

with Poisson bracket

\[ \{ \psi(x), \psi^*(y) \} = i \delta(x-y). \]  

We go over to the lattice model. We consider a one-dimensional lattice with \( N \) sites and step \( \Delta \), closed into a ring \( (N+1=N) \); the L operator of the lattice model has the form [7]

\[ L(n|\lambda) = \begin{pmatrix} 1 - \frac{\Delta \lambda}{2} + \frac{c \Delta^2}{2} & -i \Delta \phi \psi^* \rho_n \\ i \Delta \phi \rho_n \psi & 1 + \frac{\Delta \lambda}{2} - \frac{c \Delta^2}{2} \psi^* \psi \end{pmatrix}, \]  

where

\[ \rho_n = \sqrt{1 + \frac{c \Delta^2}{4}} \psi^* \psi, \]  

\( \psi_n \) being a complex Bose field on the lattice with the standard Poisson brackets

\[ \{ \psi_n, \psi_m^* \} = (i/\Delta) \delta_{nm}. \]  

The L operator (3) is intertwined with the same \( r \) matrix as in the continuous case:

\[ \{ L(n|\lambda), L(m|\mu) \} = \delta_{nm} [ L(n|\lambda), L(n|\mu) ], \]  

\[ r(\lambda, \mu) = e^{(\lambda - \mu)^{-1} \Pi}, \]  

where \( \Pi \) is the permutation matrix. The L operator (3) has the important properties

\[ \sigma_i L(n|\lambda) \sigma_i = L(n|\lambda), \quad L^T(n|\lambda) = \sigma_i L(n|\lambda) \sigma_i, \]  

\[ \det L(n|\lambda) = (\Delta/4) (\lambda - \nu) (\lambda - \bar{\nu}), \]  

\[ \nu = -2i/\Delta, \]  

where \( \sigma_i \) is a Pauli matrix. When the determinant vanishes, the L operator is transformed into the one-dimensional projection operator

\[ L_\alpha(n|\nu) = \alpha_\nu(n) \alpha_\nu(n). \]  

Here, the vector \( \alpha \) is

\[ \alpha_\nu(n) = -i \Delta \sqrt{\frac{c}{2}} \psi^*(n), \quad \alpha_{\bar{\nu}}(n) = \bar{\nu} \sqrt{\frac{c}{2}} \psi^* \psi = \bar{\nu} \rho_n. \]  

For \( \lambda = \bar{\nu} \), the L operator is also transformed into a one-dimensional projection operator (see (8)):

\[ L_\alpha(n|\nu) = (\sigma_\nu \alpha(n), \sigma_\nu \alpha(n)). \]  

We construct the monodromy matrix \( T(\lambda) \) in the standard manner:

\[ T(\lambda) = L(N|\lambda) \ldots L(2|\lambda) L(1|\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}, \]  

Its Poisson bracket is also specified by Eq. (6). The trace of the monodromy matrix,

\[ \tau(\lambda) = A(\lambda) + D(\lambda) = \text{tr} T(\lambda), \]  

generates conservation laws. We consider the logarithmic derivative of \( \tau(\lambda) \) at the point \( \nu \):

\[ \frac{\partial}{\partial \lambda} \ln \tau(\lambda) |_{\lambda=\nu} = -i \Delta \sum_{n=1}^{N} \frac{(\lambda(n+1) \sigma_\nu(n-1))}{(\sigma_\nu(n+1) \sigma_\nu(n)) (\alpha(n)\alpha(n-1))}. \]  

Here, the round brackets denote the scalar product of two two-dimensional vectors \( \langle \alpha \beta \rangle = \bar{\alpha}_1 \alpha_1 + \bar{\alpha}_2 \alpha_2. \) As the lattice Hamiltonian, we take...