DIPOLE INTERACTION OF AN OSCILLATOR
WITH SCALAR FIELD

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The dipole interaction of an oscillator with a scalar field in one-dimensional space is investigated. Solutions of the classical equations of motion are found and conditions for the classical Hamiltonian to be bounded below are obtained. In the quantum case, the choice of the zeroth approximation of perturbation theory is investigated in the case when the spectra of the free and total Hamiltonians do not coincide.

1. Introduction

In perturbation theory for small coupling constants, it is assumed that the zeroth approximation to the wave function is an eigenfunction of the free Hamiltonian. In the exactly solvable model of dipole interaction of an oscillator with a scalar field it is possible to have the case when the discrete mode corresponding to the oscillator in the free Hamiltonian is not present in the exact solution, i.e., the spectrum of the complete Hamiltonian does not coincide with the spectrum of the free Hamiltonian. In this case, the system of eigenfunctions of the free Hamiltonian is a redundant system for the construction of exact states, and it is therefore necessary to impose restrictions on the choice of the functions of the first approximation of perturbation theory.

We shall investigate this question, considering for simplicity the interaction of an oscillator with a scalar field in one-dimensional space. The generalization to the three-dimensional case is trivial and does not lead to any new conclusions. A similar problem arises, for example, in the theory of strong coupling [1].

Since the creation and annihilation operators used to diagonalize the total and the free Hamiltonian are related by a linear Bogolyubov transformation [2], the main tool for investigating the perturbation series is the generating functional of the matrix elements of this transformation obtained in [3]. We shall not prove the validity of our expansions with respect to the coupling constant, which could be the subject of a separate and rather nontrivial investigation; we restrict ourselves to answering the following question: if in this model perturbation theory is justified at all, what must be chosen as the zeroth approximation?

2. Classical Solution of the Equations of Motion

We consider the Hamiltonian that describes the dipole interaction of an oscillator with a scalar field in one-dimensional space,

\[ H = \frac{p(t)^2}{2} + \frac{\omega q(t)^2}{2} + \frac{\lambda}{2} \int dx \left\{ \pi^2(x, t) + q(x, t) \left( -\frac{\partial^2}{\partial x^2} + \mu^2 \right) q(x, t) \right\} - g q(t) \int dx \frac{dp(x)}{dx} q(x, t). \]  

(1)

In the expression (1) \( q(t), p(t), q(x, t), \pi(x, t) \) are canonical momenta and coordinates of the oscillator and the field; \( \rho(x) \) is an even function that decreases at infinity; \( g \) is the coupling constant.

We define the classical Poisson brackets by

\[ \{A, B\} = \frac{\partial A}{\partial \pi} \frac{\partial B}{\partial q} - \frac{\partial A}{\partial q} \frac{\partial B}{\partial \pi} + \int dx \left( \frac{\delta A}{\delta \varphi(x)} \frac{\delta B}{\delta \chi(x)} - \frac{\delta A}{\delta \chi(x)} \frac{\delta B}{\delta \varphi(x)} \right). \]

It follows that \( \{q(t), p(t)\} = 1, \{q(x, t), \pi(y, t)\} = \delta(x-y) \).

The equations of motion are

\[ \dot{q}(t) = \{q(t), H\}, \dot{\pi}(t) = \{\pi(t), H\} = -\omega q(t) + g \int dx \rho'(x) q(x, t), \]

(2)

\[ \dot{q}(x, t) = \{q(x, t), H\}, \dot{\pi}(x, t) = \{\pi(x, t), H\} = (\lambda - \mu^2) q(x, t) + g q(t) \rho'(x). \]

(3)
The dot denotes the time derivative. Eliminating $p(t)$ and $\pi(x, t)$, we arrive at the equations

$$\bar{q}(t) = -i\omega q(t) + \int dx \rho'(x) \varphi(x, t),$$

$$\varphi(x, t) = -(-\Delta + \mu^2) \varphi(x, t) + gq(t)\rho'(x).$$

We shall seek the solution of Eqs. (4) and (5) in the form

$$q(t) = e^{-i\omega t} q_0, \quad \varphi(x, t) = e^{-\mu x} \varphi_0(x).$$

For $q_k$ and $\varphi_k(x)$ we obtain the equations

$$\omega^2(k) q_k = -\omega q_k - g \int dx \rho'(x) \varphi_k(x),$$

$$\omega^2(k) \varphi_k(x) = -(-\Delta + \mu^2) \varphi_k(x) - g\rho'(x) \varphi_k.$$  

The general solution of Eq. (7) is

$$\varphi_k(x) = C(k) e^{\omega x} + q_k \int dy G(k, x-y) \rho'(y),$$

where $\omega^2(k) = k^2 + \mu^2$, and $G$ is one of the Green’s functions defined by $(\Delta + k^2) G(k, x-y) = \delta(x-y)$, its explicit form being determined by the asymptotic behavior of the solutions.

Substituting (8) in (6), we arrive at the relation

$$\omega^2(k) q_k = gC(k) (-ik) \rho(k),$$

where $\rho(k) = \int e^{-ikx} \rho(x) dx$, $\Delta^<(k^2) = \omega^2 - \omega^2(k) - \frac{g^2}{2\pi} \int \frac{dl P^>(l)}{l^2 + k^2 + i\varepsilon}.$

We have used the $G^\pm$ Green’s functions

$$G^\pm(k, x-y) = \frac{1}{2\pi} \int \frac{dl e^{(l)(x-y)}}{l^2 + k^2 + i\varepsilon}.$$

The Green’s functions $G^\pm$ give asymptotic behavior at infinity corresponding to converging or diverging waves.

The solutions with $C(k) = 0$ corresponding to the discrete mode are determined by the condition

$$\Delta^< (k^2) = 0.$$ (9)

We define the function $\Delta(z)$, which is analytic in the complex plane with the cut $[0, \infty)$:

$$\Delta(z) = \omega^2 - \mu^2 - z - \frac{g^2}{2\pi} \int \frac{dl P^>(l)}{l^2 - z}.$$  

Then $\Delta^<(k^2)$ are the boundary values of the function $\Delta(z)$ as the real axis is approached from below and above:

$$\Delta^<(k^2) = \lim_{\varepsilon \to 0} \Delta(k^2 \pm i\varepsilon).$$

We set $z = a + ib$, and then

$$\Delta(z) = \omega^2 - \mu^2 - a - \frac{g^2}{2\pi} \int \frac{dl P^>(l)}{(l^2 - a)^2 + b^2} - ib \left( \frac{g^2}{2\pi} \int \frac{dl P^>(l)}{(l^2 - a)^2 + b^2} \right).$$

It follows from this expression that the zeros of the function $\Delta(z)$ are to be sought only for $b = 0$, i.e., on the real axis. It is easy to show that for $k^2 > 0$ the imaginary part of the function $\Delta^<(k^2)$ cannot vanish, i.e., roots of the equation $\Delta^<(k^2) = 0$ can exist only for $k^2 < 0$. For $k^2 < 0$, we obtain the equation

$$\omega^2 - \mu^2 = \frac{g^2}{2\pi} \int \frac{dl P^>(l)}{l^2 - k^2}.$$  

Figure 1 shows graphs of the functions $\omega^2 - \mu^2 - k^2$ and $\frac{g^2}{2\pi} \int \frac{dl P^>(l)}{l^2 - k^2}$ (the monotonicity of the latter is obvious) (in Fig. 1, the dashed line corresponds to $\omega^2 > \mu^2$, and the continuous line to $\omega^2 < \mu^2$).

It is easy to see that for

$$\omega^2 > \frac{g^2}{2\pi} \int \frac{dl P^>(l)}{\omega^2(l)}.$$  

(10)