A relativistic string is described by the inverse scattering method for periodic potentials. The auxiliary spectral problem and the analytic properties of the Bloch functions are investigated. A complete set of involutive relativistically invariant integrals of the motion is obtained. Equations are obtained for string configurations that have a finite number of gaps in the spectrum of the auxiliary problem. It is shown that such a condition is equivalent to stationarity of the configurations with respect to the higher string equations. The quantum theory in the single-gap sector is discussed.

In the present paper, we propose a formalism for describing a relativistic string by means of the inverse scattering method, or rather its modification for periodic potentials developed by Novikov, Dubrovin, Matveev, and Its [1-3]. The use of this method made it possible to find a regular procedure for reduction of the string to systems with a finite number of degrees of freedom. The first example of such reduction was constructed in [4] without recourse to the inverse scattering method by simple "freezing" of infinitely many degrees of freedom. The resulting system with a finite number of degrees of freedom preserves the characteristic features of a relativistic string and admits a consistent quantum description in a space of any number of dimensions. Despite the finite number of degrees of freedom, the reduced system has a rich spectrum of states and Regge trajectories, these being nonlinear, in contrast to traditional trajectories. It should be recalled that the currently existing methods of quantization, which are discussed, for example, in the review [5], do not lead to a consistent string theory in a space with arbitrary number of dimensions. In the present paper, we consider only the case of four-dimensional space-time, although in principle similar constructions are possible for a larger number of dimensions.

An undoubted advantage of the proposed approach is its relativistic and gauge invariance. For example, we have succeeded in constructing a complete set of involutive relativistically invariant integrals of the motion. These functions generate Hamiltonian flows on the string phase space, the stationarity points of these flows forming finite-dimensional orbits of the gauge group.

The similar flows in the Korteweg–de Vries (KdV) theory are called the higher KdV equations. Introducing the concept of higher string equations, one can show that their stationary solutions form a finite-dimensional space corresponding to a gauge-invariant reduction of the string phase space.

Our formalism is based on study of the auxiliary spectral problem for a matrix system of first order. A similar system, but with fewer components of the potential occurs in the theory periodic solutions of the nonlinear Schrödinger equation [3]. In the paper, we derive and discuss the main properties of the Bloch functions and the spectrum needed for the introduction and formulation of the conditions for a finite-gap string configuration. We show that this condition is equivalent to the condition of stationarity with respect to one of the higher string equations. In conclusion, we discuss string quantization in the finite-gap sector.

1. Preliminary Remarks

A relativistic string in four-dimensional space–time is specified by means of a set of functions of two parameters: \( x_\mu = x_\mu(\sigma, \tau), \mu = 0, 1, 2, 3. \) The parameter \( \tau \) describes the evolution of the string, whereas \( \sigma \) labels its points.

We consider a closed string, and it is therefore convenient to take the domain of variation of the parameter \( \sigma \) from 0 to 2\( \pi \), with \( x_\mu(0, \tau) = x_\mu(2\pi, \tau) \). It is also convenient to assume that the functions \( x_\mu(\sigma, \tau) \) and continued periodically to all the remaining values of \( \sigma \).
Following [6], we choose the action for the string in the form

$$S = \frac{1}{2\pi\alpha'} \int d\tau \int_0^{2\pi} d\sigma' \sqrt{(\dot{x}'(\sigma'))^2 - g_\sigma''^2},$$

(1)

where the dot and the prime denote the derivatives with respect to \(\tau\) and \(\sigma\), respectively. For simplicity, we shall assume that the dimensional factor \(2\pi\alpha'\), which ensures that the action has zero dimension, is 1.

It is well known that the action (1) is parametrically invariant, as a consequence of which the canonical Hamiltonian vanishes, and in accordance with Dirac's theory [7] the role of the Hamiltonian is played by a linear combination of the constraints:

$$\chi_i(\sigma, \tau) = \frac{1}{2} \left(p_i(\sigma, \tau) + x'_i(\sigma, \tau)\right), \quad \chi_2(\sigma, \tau) = p_2(\sigma, \tau) x'_2(\sigma, \tau).$$

The canonical momenta that occur in these expressions are defined by the equation

$$p_\mu(\sigma, \tau) = -\delta S/\delta \dot{x}_\mu(\sigma, \tau).$$

The canonical Poisson brackets for the coordinates and momenta of the closed string have the form

$$\{x_\mu(\sigma, \tau), p_\nu(\sigma', \tau')\} = \eta_{\mu\nu} \Delta(\sigma - \sigma'),$$

(2)

where \(\Delta(\sigma)\) is the periodic \(\delta\) function

$$\Delta(\sigma) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{in\sigma}.$$

With respect to the brackets (2), the constraints \(\chi_i(\sigma, \tau)\) and \(\chi_2(\sigma, \tau)\) form a constraint algebra of the first class [7]. A detailed discussion of this algebra and the corresponding group is contained in [8].

Introducing the notation

$$a_\mu(\sigma, \tau) = p_\mu(\sigma, \tau) + x'_\mu(\sigma, \tau), \quad b_\mu(\sigma, \tau) = p_\mu(\sigma, \tau) - x'_\mu(\sigma, \tau),$$

it is convenient to go over to linear combinations of the constraints \(\chi_1\) and \(\chi_2\):

$$L(\sigma, \tau) = -\frac{1}{2} a^1(\sigma, \tau), \quad \bar{L}(\sigma, \tau) = \frac{1}{2} b^1(\sigma, \tau).$$

(3)

Using (2), we can show that the variables \(a_\mu(\sigma, \tau)\) and \(b_\mu(\sigma, \tau)\) and the constraints (3) satisfy the relations

$$\{a_\mu(\sigma, \tau), a_\nu(\sigma', \tau')\} = -\{b_\mu(\sigma, \tau), b_\nu(\sigma', \tau')\} = 2\eta_{\mu\nu} \Lambda(\sigma - \sigma'), \quad \{a_\mu(\sigma, \tau), b_\nu(\sigma', \tau)\} = 0,$$

$$\{L(\sigma, \tau), L(\sigma', \tau')\} = -\left(L(\sigma, \tau) + L(\sigma', \tau')\right) \Lambda(\sigma - \sigma'), \quad \{L(\sigma, \tau), \bar{L}(\sigma', \tau')\} = -\left(L(\sigma, \tau) + \bar{L}(\sigma', \tau')\right) \Lambda(\sigma - \sigma'),$$

$$\{L(\sigma, \tau), \bar{L}(\sigma', \tau)\} = 0.$$

The generators of the Poincaré group can be obtained by means of Noether's theorem in the standard manner and have the form

$$P_\mu = \int_0^{2\pi} d\sigma p_\mu(\sigma, \tau) = \int_0^{2\pi} d\sigma a_\mu(\sigma, \tau) = \int_0^{2\pi} d\sigma b_\mu(\sigma, \tau),$$

(4)

$$M_{\mu v} = \int_0^{2\pi} d\sigma \left(x_\mu(\sigma, \tau) p_\nu(\sigma, \tau) - x_\nu(\sigma, \tau) p_\mu(\sigma, \tau)\right).$$

(5)

From \(P_\mu\) and \(M_{\mu v}\) we can construct the Pauli–Lubański pseudovector \(W_\mu = \frac{1}{2} \varepsilon_{\mu
u\rho} P_\nu M_{\rho\nu}\) and the spin tensor*

$$S_{\mu\nu} = M_{\mu v} - \frac{1}{P^2} \left(M_{\mu\nu} P^\rho P_\rho - M_{\rho\nu} P^\rho P_\rho\right) = \frac{1}{P^2} \varepsilon_{\mu
u\rho} P^\rho W^\rho.$$

(6)

The momentum (4) and the angular momentum (5) form with respect to the Poisson bracket the ordinary Lie algebra of the Poincaré group. As was noted in [4], for a closed relativistic string there exists a decomposition of \(S_{\mu\nu}\) into a sum of two integrals of the motion. We introduce the vectors

$$A_2 = -\frac{1}{4\sqrt{1-P^2}} \varepsilon_{\mu
u\rho} P^\rho \int_0^{2\pi} d\sigma a^{\rho'}(\sigma) \int_0^{2\pi} d\sigma a^{\rho'}(\sigma'), \quad B_2 = \frac{1}{4\sqrt{1-P^2}} \varepsilon_{\mu
u\rho} P^\rho \int_0^{2\pi} d\sigma b^{\rho'}(\sigma) \int_0^{2\pi} d\sigma b^{\rho'}(\sigma').$$

We assume that \(P^2 \neq 0\); the case \(P^2 = 0\) must be considered separately.