QUANTUM GROUPS, $q$ OSCILLATORS, AND COVARIANT ALGEBRAS

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The physical interpretation of the basic concepts of the theory of covariant groups — coproducts, representations and corepresentations, action and coaction — is discussed for the examples of the simplest $q$ deformed objects (quantum groups and algebras, $q$ oscillators, and comodule algebras). It is shown that the reduction of the covariant algebra of quantum second-rank tensors includes the algebras of the $q$ oscillator and quantum sphere. A special case of covariant algebra corresponds to the braid group in a space with nontrivial topology.

In memory of Mikhail Konstantinovich Polivanov

The formalism of the quantum inverse scattering method (R-matrix approach), which was used in [1] to formulate the theory of quantum Lie groups and algebras, has stimulated a considerable growth of interest in these new mathematical objects and their use by theoretical physicists. Convincing examples have been given of the description of the symmetry properties of physical models by means of quantum groups and algebras (see, for example, [2–6]), and this has been done both in the normal context, in which the Hamiltonian $H$ of the model commutes with the generators $X_i$ of the quantum algebra, $[H, X_i] = 0$, as well as in the more complicated situation of conformal field theory. Quantum groups and algebras are also the basis for new approaches to the possible structure of spacetime at short distances (see [7,8]), permitting a natural definition of homogeneous quantum spaces.

However, despite the intensive development of the mathematical theory of quantum groups (see [9]), the physical interpretation of many results in this field warrant, in our view, greater attention. As always in such situations, simple examples are the easiest to discuss. We shall restrict ourselves to ones that are well known: the quantum group $F_q(GL(2))$, the quantum algebra $sl_q(2)$, the $q$-deformed oscillator, and a new example [10] of a very simple quantum algebra $A$ that is generated by the reflection equation and is covariant with respect to the quantum group $F_q(GL(2))$. We shall also discuss concepts such as coproduct, representation and corepresentation, action and coaction.

The quantum group $F_q(GL(2))$ is generated as an associative algebra by four generators $a, b, c, d$ that satisfy

$$ab = qba, \quad ac = qca, \quad [a, d] = \omega bc, \quad bd = qdb, \quad cd = qdc, \quad [b, c] = 0,$$

where $q$ is the complex deformation parameter, and $\omega = q - q^{-1}$. The relations (1) define multiplication in the algebra $F_q$. They can be expressed in a compact form [1] by using the $2\times2$ matrix

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2)$$

and a $4\times4$ matrix $R$ with diagonal $(q, 1, 1, q)$ and a single nonvanishing element $R_{21,12} = \omega$ below the diagonal,

$$RT_1T_2 = T_1T_2R, \quad (3)$$

where $T_1 = T \otimes I$, $T_2 = I \otimes T$. The rows and columns of the $R$ matrix in $\mathbb{C}^2 \otimes \mathbb{C}^2$ are labeled by the index pairs $11, 12, 21, 22$. As a mathematical object, $F_q$ is a Hopf algebra, which means that besides the multiplication $\mu$ there are three further operations present [1]: the coproduct $\Delta$: $F_q \to F_q \otimes F_q$, the antipode $s$: $F_q \to F_q$, and coidentity $\varepsilon$: $F_q \to \mathbb{C}$. These three mappings are defined on the generators and are extended to the complete algebra $F_q$ by the requirement that $\Delta$ and $\varepsilon$ be homomorphisms and $s$ an antihomomorphism $[s(a, b) = s(b)s(a)]$. They can also be conveniently expressed in the matrix form

$$\Delta(T) = T(S)T, \quad \Delta(t_{ij}) = \sum_k t_{ik} \otimes t_{kj}, \quad s(T)T = Ts(T) = I, \quad \varepsilon(T) = I, \quad (4)$$

where, as above, $I$ is the $2\times2$ unit matrix. The operations $\mu, \Delta, \varepsilon, s$ are related by some axioms [1], the validity of which is almost obvious in the $R$-matrix approach.
The quantum algebra \( \mathfrak{sl}_q(2) \) is generated by three generators \( J, X_+, X_- \), which satisfy the commutation relations

\[
[J, X_\pm] = \pm X_\pm, \quad [X_+, X_-] = \frac{q^{2J} - q^{-2J}}{q - q^{-1}} = [2J],
\]

where we have introduced the notation \( [x]_q = (q^x - q^{-x})/(q - q^{-1}) \). The three additional mappings that define the structure of a (quasitriangular [11]) Hopf algebra are specified by

\[
\begin{align*}
\Delta J &= J \otimes 1 + 1 \otimes J, \\
\Delta(X_\pm) &= X_\mp \otimes q^{-J} + q^J \otimes X_\pm, \\
\varepsilon(J) &= \varepsilon(X_\pm) = 0,
\end{align*}
\]

The relations (5) and (6) can also be expressed in \( R \)-matrix form by using the upper and lower triangular matrices \( L^\pm \) [11,12].

The associative algebra \( A(q) \) of the deformed oscillator can be defined even more simply, since, like \( \mathfrak{sl}_q(2) \), it is generated by three generators \( \alpha, \alpha^+, N \) satisfying

\[
[N, \alpha] = -\alpha, \quad [N, \alpha^+] = \alpha^+, \quad [\alpha, \alpha^+] = q^{-2N},
\]

but the structure of the Hopf algebra for it is unknown.

Finally, the associative algebra \( A \) associated with the reflection equation [10] is generated by four generators \( \alpha, \beta, \gamma, \delta \) that satisfy the quadratic relations (where \( \omega = q - q^{-1} \))

\[
\begin{align*}
[\alpha, \beta] &= \omega \alpha \gamma, \\
[\alpha \gamma] &= q^2 \gamma \alpha, \\
[\alpha, \delta] &= \omega (\beta \gamma + \gamma \delta), \\
[\beta, \gamma] &= 0, \\
[\beta, \delta] &= \omega \gamma \delta, \\
[\gamma \delta] &= q^2 \gamma \delta.
\end{align*}
\]

The structure of the Hopf algebra for it is also unknown.

The \( R \)-matrix form of expression (8) uses the \( 2 \times 2 \) matrix \( K \) of the generators \( \alpha, \beta, \gamma, \delta \), and the \( R \) matrix of the quantum group \( F_q(GL(2)) \) (3),

\[
R K_1 R^t K_2 = K_1 R^t K_1 R,
\]

where the matrix \( R^t \) is transposed with respect to the first space in \( C^2 \otimes C^2 \). Therefore, the diagonal in \( R^t \) is the same as in \( R \), and the single nonvanishing nondiagonal element \( \omega \) is in the top right corner.

If the above algebras are used as algebras of observables, then it is necessary to introduce in them a real structure (\( * \) operation). For example, from \( \mathfrak{sl}_q(2, \mathbb{C}) \) we obtain \( \mathfrak{su}_q(2), \mathfrak{su}_q(1, 1), \mathfrak{sl}_q(2, \mathbb{R}) \) with appropriate restrictions on the deformation parameter \( (q \in \mathbb{R}, \ |q| = 1) \). However, in this paper we shall not dwell on the question of real forms.

The main aim of this paper is to note intimate connections between these algebras, some of which are well known, for example, the duality of \( \mathfrak{sl}_q(2) \) and \( F_q(SL(2)) \) [1], and to propose a physical interpretation of these connections. We note also that the above associative algebras have central elements:

\[
c_1(q) = \alpha^+ \alpha - [N; q^{-2}], \quad c_2 = X_- X_+ + [J]_q [J + 1]_q; \\
c_3(q) = \alpha^+ \alpha - [N; q^{-2}], \quad [n; q] = (q^n - 1)/(q - 1); \\
z_1 = \beta - q \gamma, \quad z_2 = \alpha \delta - q^2 \beta \gamma.
\]

Note that the matrix \( K \) satisfies a characteristic equation

\[
K \varepsilon K = z_2 \varepsilon_q - qz_1 K,
\]

where \( \varepsilon_q \) is the quantum metric for \( F_q(SL(2)) \).

Before we discuss representations, we shall dwell on the connections between the algebras, since we are concerned with associative algebras. The deformed oscillator can be obtained from the quantum algebra \( \mathfrak{sl}_q(2) \) by contraction [13] (\( f \rightarrow 0 \)),

\[
\alpha = \lim f^{1/3} \alpha, \quad \alpha^+ = \lim f^{-1/3} \alpha^+, \quad q^{-N} = \lim f^{N},
\]

in particular, \( \lim f^2 \omega c_2 = c_2(q) + q^2/(q^2 - 1) \). The remaining relations (6) of the Hopf algebra \( \mathfrak{sl}_q(2) \) do not survive in this limit. One can nevertheless obtain finite expressions on the right-hand side for the coproduct \( \Delta \),

\[
\psi(\alpha) = \alpha \otimes q^{-J} + \omega^{1/2} q^{-N} \otimes X_+, \quad \psi(N) = N \otimes 1 - 1 \otimes J
\]

and interpret it as a mapping \( \psi \) from the algebra \( A(q) \) to the tensor product \( A(q) \otimes \mathfrak{sl}_q(2) \) that preserves the commutation relations (7) of the algebra \( A(q) \) of the \( q \) oscillator:

\[
[\psi(\alpha), \psi(\alpha^+)] = q^{-2\psi(N)}, \quad [\psi(N), \psi(\alpha)] = \psi(\alpha).
\]

Such a mapping, satisfying the properties of consistency with the coproduct, \( (\psi \otimes \text{id}) \circ \psi = (\text{id} \otimes \Delta) \circ \psi \), and coidentity,