ABSTRACT. The solutions of elliptic regularizations of the heat equation converge to the solutions of the corresponding evolution equation in weak energy norms. The technique is of relevance in control theory as well as variational inequalities. We show that the convergence actually takes place in some Hölder space. The result holds for a class of parabolic equations in divergence form and sufficiently integrable forcing terms.

SOMMARIO. Le soluzioni delle regolarizzanti ellittiche dell’equazione del calore, convergono alle soluzioni della corrispondente equazione di evoluzione in norme deboli dell’energia. Questa tecnica trova applicazione nelle teorie dei controlli e disuguaglianze variazionali. Si dimostra chela convergenza è forte ed ha luogo in norme Hölderiane. Il risultato si estende ad una classe di equazioni paraboliche in forma di divergenza con termini forzanti sufficientemente integrabili.

KEY WORDS. Free boundary problems, Elliptic regularizations, Parabolic equations.

1. INTRODUCTION AND RESULTS

The elliptic regularizations
\[ H_\varepsilon(u_\varepsilon) = u_{\varepsilon,t} - \varepsilon u_{\varepsilon,xx} - \Delta u_\varepsilon = f \quad \text{in } \Omega_\varepsilon, \]
converge to a solution of the heat equation in weak energy norms. One of the results of this note is that the convergence actually takes place in \( C^\alpha(\Omega_\varepsilon) \) for some \( \varepsilon > 0 \).

The technique of elliptic regularization has been used, for instance in [1] to prove the existence of solutions to certain variational inequalities, as well as in control theory.

Let \( \mathcal{L} \) be the elliptic operator
\[ \mathcal{L}(v) = (a_{ij} v_{x_j} + b_i v)_{x_i} + c_i v_{x_i} + d v, \]
where the symmetric matrix \((a_{ij})\) satisfies
\[ \lambda^{-1} |\xi|^2 \leq a_{ij} \xi_i \xi_j \leq \lambda |\xi|^2, \]
\[ \forall \xi \in \mathbb{R}^N \text{ for some } \lambda \geq 1. \]

Let \( \mathcal{X} \subseteq \Omega_\varepsilon \) be a compact set. We assume that there exist positive numbers \( a, b, C_a \) and \( C_b \) so that for each pair of points \((x_1, t_1), (x_2, t_2) \in \mathcal{X}, \)
\[ |a_{ij}(x_1, t_1) - a_{ij}(x_2, t_2)| \leq C_a(|x_1 - x_2|^\alpha + |t_1 - t_2|), \]
\[ i, j = 1, 2, \ldots, N, \]
and
\[ |b_i(x_1, t_1) - b_i(x_2, t_2)| \leq C_b(|x_1 - x_2|^\beta + |t_1 - t_2|), \]
\[ i = 1, 2, \ldots, N; \]
while the remaining lower order coefficients are locally bounded in \( \mathcal{X}. \)

Consider the elliptic regularizations
\[ \begin{align*}
\{ v_{\varepsilon} \in & L^2_{\text{loc}}(0, T; L^2_{\text{loc}}(\Omega)) \cap L^2(0, T; W^{1,2}_{\text{loc}}(\Omega)) ; \\
& v_{\varepsilon,t} - \varepsilon v_{\varepsilon,xx} - \mathcal{L}(v_{\varepsilon}) = f \quad \text{weakly in } \Omega_\varepsilon, \end{align*} \]

where
\[ f \in L^p_\text{loc}(\Omega_T), \quad p > \frac{2(N + 1)}{2a - 1}. \]

We assume that the regularizations (1.5) have locally finite energy, uniformly in \( \varepsilon \), that is for every compact subset \( \mathcal{X} \) of \( \Omega_\varepsilon \) the quantity
\[ \mathcal{E}(\mathcal{X}) \equiv \| v_{\varepsilon} \|_{L^2} + \| Dv_{\varepsilon} \|_{L^2}, \]
is bounded uniformly in \( \varepsilon \) and possibly depending upon \( \mathcal{X} \). We say a constant \( \gamma \equiv \gamma(\mathcal{X}) \) depends only upon the data on \( \mathcal{X} \) if it can be determined \( a \) priori only in terms of \( \mathcal{X}, N, A, p, a, b, C_a, C_b \) and the quantities
\[ \| v_{\varepsilon} \|_{L^\infty}, \| Dv_{\varepsilon} \|_{L^\infty}, \| f \|_{L^2}. \]

THEOREM. There exists a constant \( \alpha \in (0, 1) \) that can be determined \( a \) priori only in terms of \( N, a, b \) and \( p \), such that for every compact subset \( \mathcal{X} \) of \( \Omega_\varepsilon \), there holds
\[ \| u_{\varepsilon} \|_{C^\alpha(\mathcal{X})} \leq \gamma(\mathcal{X}) \quad \text{uniformly in } \varepsilon. \]

REMARK. In (1.5) we may allow the coefficients \( a_{ij}, b_i, c_i \), \( d \) and the function \( f \) to depend on the regularizing parameter \( \varepsilon \), provided their required bounds are independent of \( \varepsilon \).

2. THE HEAT EQUATION

To illustrate our ideas, we first prove the theorem in the context of the heat equation. Let \( \nu \) be a solution of
\[ v_t - \nu_{xx} - \Delta v = h \quad \text{in } \Omega_T. \]

Fix \( \mathcal{X} \subset \Omega_T \) and proceed to derive \( \varepsilon \)-independent estimates for the classical solutions of
\[ u_t - \varepsilon u_{xx} - \Delta u = 0 \quad \text{in } \mathcal{X}. \]

Fix a point \((x_0, t_0) \in \mathcal{X} \), assume up to a translation that it
coincides with the origin, and for \( \eta, \rho > 0 \), construct the cubes \( K_{\eta \rho} \) and the cylindrical domains \( Q_{\eta \rho} \) defined by

\[
K_{\eta \rho} = \max_{1 \leq i \leq N} \{ |x_i| < \eta \rho \},
\]

\[
Q_{\eta \rho} = K_{\eta \rho} \times \{ -\eta^2 \rho < t < \eta^2 \rho \}.
\]

If \( \rho = 1 \) we let \( K_1 \equiv K \) and \( Q_1 \equiv Q \). For a non-negative integer \( n \) we denote with \( D^n v \) a generic derivative of \( v \) of order \( n \) with respect to the space variables.

**LEMMA 1.** Let \( p \) be any number so that \( Q_2^p \subset X \). If \( \sqrt{\varepsilon} \leq \rho \), then there exists a constant \( C \) independent of \( \varepsilon \) and \( p \) so that for every pair of non-negative integers \( k, l \leq N + 1 \), there holds:

\[
\left\langle D^k \left( D^l u \right) \right\rangle_{Q_{2^p}} \leq C \left\langle D^k \left( D^l u \right) \right\rangle_{Q_{2^p}}.
\]

\[
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\]

Proof. Let \( \zeta \) be a smooth cutoff function in \( Q_{2^p} \), \( \zeta = 1 \) on \( Q_p \), \( \left| \nabla \zeta \right| \leq C/p \) and \( \left| \zeta \right| \leq C/p^2 \). The function \( w = D^n \left( D^n u \right) \) satisfies

\[
w_t - c w_{tt} + D^n \left( D^n u \right) = 0 \quad \text{in} \quad Q_{2^p}.
\]

Multiply (2.5) by \( w \zeta^2 \), and integrate over \( Q_{2^p} \) to get

\[
\int_{Q_{2^p}} (2w^2 \zeta^2 + \left| \nabla w \right|^2 \zeta^2) \, dx \, dt \leq C \int_{Q_{2^p}} \left| \nabla w \right|^2 \, dx \, dt.
\]

\[
\left| \nabla w \right|_{L^2} \leq C \left| \nabla w \right|_{L^2}.
\]

Since \( \varepsilon \leq \rho^3 \), (2.3) follows.

To prove (2.4), multiply (2.5) by \( w_i \zeta^2 \), and integrate over \( Q_{2^p} \). We obtain

\[
\int_{Q_{2^p}} \left( w_i \zeta^2 \right) \, dx \, dt \leq C \int_{Q_{2^p}} \left( \left| w_i \right|^2 + \left| \nabla w \right|^2 \right) \, dx \, dt.
\]

Rescaling and using (2.6), the result follows. \( \square \)

**LEMMA 2.** Let \( R \) be any positive number so that \( Q_R \subset X \). Then for all \( \sqrt{\varepsilon} \leq \rho \leq R/2 \), there is a constant \( C \) independent of \( \varepsilon \), \( p \) and \( R \) so that:

\[
\int_{Q_{R^2}} |D^k w|^2 \, dx \, dt \leq C \int_{Q_{R^2}} |D^k w|^2 \, dx \, dt,
\]

\[
\int_{Q_{R^2}} |D^k u|^2 \, dx \, dt \leq C \int_{Q_{R^2}} |D^k u|^2 \, dx \, dt.
\]

Proof. Assume that \( \sqrt{\varepsilon} \leq \rho \leq R/2^{N+1} \). Let \( \zeta \) be a cutoff function in \( Q_{R^{2^{N+1}}} \), where \( \zeta = 1 \) on \( Q_p \), \( |D^k \zeta| \leq C/R^k \), and \( |\zeta| \leq C/R^2 \). Then

\[
\int_{Q_{R^2}} |D^k w|^2 \, dx \, dt \leq C \rho^{N+2} \|D^k w\|^2_{L^2(Q_{R^2}, Q_{R^2})},
\]

\[
\int_{Q_{R^2}} |D^k u|^2 \, dx \, dt \leq C \rho^{N+2} \|D^k u\|^2_{L^2(Q_{R^2}, Q_{R^2})}.
\]

For \( (x, t) \in Q_{R^{2^{N+1}}} \), we have

\[
\left( \frac{D^N}{\partial t^2} (\zeta^2 w) \right) = \int_{-R^N/4}^t \frac{\partial}{\partial t} (\zeta^2 w) \, dt
\]

\[
\leq C \int_{Q_{2^{N+1}}} \left| D^N \frac{\partial}{\partial t} (\zeta^2 w) \right| \, dx.
\]

Thus,

\[
\left| \zeta^2 w \right|_{L^2} \leq CR^{N+2} \int_{Q_{2^{N+1}}} \left| D^N \frac{\partial}{\partial t} (\zeta^2 w) \right| \, dx.
\]

Estimating the derivatives on the right-hand side using Lemma 1, we find that

\[
\int_{Q_{R^2}} |D^k w|^2 \, dx \, dt \leq C \rho^{N+2} \int_{Q_{R^2}} |D^k u|^2 \, dx \, dt,
\]

and thus (2.7) follows after scaling. The proof of (2.8) is similar, using \( u_t \) instead of \( w_t \). \( \square \)

**LEMMA 3.** Let \( 0 < \rho < 1 \), \( \rho_0 \geq 2 \). Then there exists a constant \( C \) independent of \( \varepsilon \), \( p \) and \( R \) so that:

\[
\int_{Q_{R^2}} |D^k w|^2 \, dx \, dt \leq C \int_{Q_{R^2}} |D^k u|^2 \, dx \, dt,
\]

\[
\int_{Q_{R^2}} |u_t|^2 \, dx \, dt \leq C \int_{Q_{R^2}} |u_t|^2 \, dx \, dt.
\]

Proof. Set \( \tau = e^{-\varepsilon t} \), \( \zeta = x/\sqrt{\varepsilon} \). Then our equation becomes

\[
\frac{\partial}{\partial t} (\tau^2 u_t) + \Delta u = 0.
\]

Set

\[
D_\rho \equiv \max_{1 \leq i \leq N} \left| \frac{\partial}{\partial x_i} \right| < \rho \times \{ 1 - \rho < |x| < 1 + \rho \}.
\]

Assume that \( \rho \leq \frac{1}{2} \). Then \( \tau^2 \in [\frac{1}{4}, \frac{2}{3}] \), and thus (2.11) is uniformly elliptic. Since \( w = u_\rho \) satisfies (2.11), by standard elliptic theory

\[
\|w\|_{L^p(D_\rho, D_\rho)} \leq C \left( \int_{D_\rho} \frac{\left| w \right|^2 \, d\zeta}{\rho} \right)^{1/2},
\]

\[
\|D_\rho u\|_{L^p(D_\rho, D_\rho)} \leq C \left( \frac{1}{\rho^{N+1}} \int_{D_\rho} \left| D_\rho u \right|^2 \, d\zeta \right)^{1/2}.
\]

Set

\[
\tilde{Q}_{\varepsilon, R} \equiv \left\{ \max_{1 \leq i \leq N} \left| x_i \right| < \sqrt{\varepsilon \rho} \times \left\{ \varepsilon \log \frac{1}{1 + \rho} < t < \varepsilon \log \frac{1}{1 - \rho} \right\} \right\}
\]

\[
\tilde{Q}_{\varepsilon, R} \equiv \left\{ \max_{1 \leq i \leq N} \left| x_i \right| < \sqrt{\varepsilon \rho} \times \left\{ \varepsilon \log \frac{1}{1 + \rho} < t < \varepsilon \log \frac{1}{1 - \rho} \right\} \right\}
\]