A Twist on Chiral Potts

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We show that the chiral Potts model may be formulated so that the rapidity lines carry a second integer variable—an increment or “twist” in each bond crossing it. This modification does not affect those properties of the chiral Potts model which lead to integrability, since it is equivalent to one of the automorphisms allowed for in the theory. In particular, transfer matrices still form commuting families and still satisfy hierarchies of functional equations. Surprisingly, the superintegrable case with twists retains the special algebraic properties which lead to its Ising-like spectra. The formalism should be useful for considering systems with twisted boundary conditions or embedded interfaces.

KEY WORDS: Boundary conditions; chiral Potts; exact solution; statistical mechanics.

1. INTRODUCTION

Recently much progress has been made in the solution of the chiral Potts model. This is a class of $N$-state two-dimensional lattice models for which solutions of the star-triangle relation$^{(1)}$ have been found. It is $Z_N$ symmetric, but chirally asymmetric: that is, the Boltzmann weight for an adjoining pair of spins $n, n'$ depends only on the difference $n - n'$, but is asymmetric under interchange of $n, n' \mod N$. The star-triangle property implies, as an immediate corollary, that there are commuting families of transfer matrices parametrized by some “rapidity” variables and also a “temperature-like” variable. Further, each commuting family generates an infinite sequence of conserved quantities, the simplest of which is an $N$-state spin chain Hamiltonian involving only nearest neighbor interactions. For a special choice of rapidity variables—the “superintegrable”

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case—there is an underlying Lie algebra identical with the algebra whereby Onsager solved the Ising model,\(^{(2)}\) and this has led to many new results in this case.\(^{(3-7)}\)

We recall some basic results for the chiral Potts model.\(^{(1,8)}\) The parametrization employs algebraic curves satisfying the homogeneous equations

\[
a_p^N + k' b_p^N = k a_p^N, \quad k' a_p^N + b_p^N = k c_p^N, \quad k^2 + k'^2 = 1 \tag{1}
\]

There are two types of interaction between neighboring spins, each labeled by a pair of rapidities—see Fig. 1. The weights are defined by

\[
\frac{W_{pq}(n)}{W_{pq}(0)} = \prod_{k=1}^{n} \frac{(d_p b_q - a_p c_q \omega^k)(b_p d_q - c_p a_q \omega^k)}{(d_p b_q - a_p c_q \omega^k)(b_p d_q - c_p a_q \omega^k)} \tag{2}
\]

\[
\frac{\overline{W}_{pq}(n)}{\overline{W}_{pq}(0)} = \prod_{k=1}^{n} \frac{(\omega a_p d_q - d_p a_q \omega^k)(c_p b_q - b_p c_q \omega^k)}{(c_p b_q - b_p c_q \omega^k)}
\]

where \(\omega = \exp(-2\pi i/N)\). The convention of Fig. 1, including the chirality which is implicit in it, is most important in what follows. For \(W_{pq}(n)\) the arrow on the bond points to the right of the rapidity \(p\), while \(q\) points to the left of \(p\). For \(\overline{W}_{pq}(n)\) both arrows point to the left of \(p\). Thus, there is no ambiguity in assigning spins and Boltzmann weights once the rapidity lines are given, and there will be no ambiguity in assigning “twists” to bonds. There are various automorphisms of the algebraic curves (1), notably

\[
R: \quad a_{Rq}, b_{Rq}, c_{Rq}, d_{Rq} = b_q, \omega a_q, d_q, c_q
\]

\[
T: \quad a_{Tq}, b_{Tq}, c_{Tq}, d_{Tq} = \omega a_q, b_q, \omega c_q, d_q \tag{3}
\]

\[
U: \quad a_{Uq}, b_{Uq}, c_{Uq}, d_{Uq} = \omega a_q, b_q, c_q, d_q
\]

![Fig. 1. Boltzmann weights for the chiral Potts model.](image)