THE BOUNDARY LAYER ON THE FREE SURFACE OF A HOLLOW VORTEX

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Izv. AN SSSR, Mekhanika Zhidkosti i Gaza, Vol. 1, No. 6, pp. 45-49, 1966

Several problems are known which are associated with the circular motion of a viscous incompressible fluid with a rotating cylinder [1, 2]. In the present paper we consider the case of unsteady circular motion of a viscous fluid with a cavity in the fluid.

Assume a tangential velocity field $V_\varphi = \Gamma/2\pi r$ ($\Gamma$ is the circulation) is created in an infinite space filled with a fluid, and near the axis there is a circular cylindrical discontinuity cavity of radius $r_0$.

![Fig. 1](image)

The mathematical model of the flow of an ideal fluid with the formation of a hollow vortex is well known [3]. Let us clarify the changes in this motion associated with the fluid viscosity.

To do this, at the time $t = 0$ we "switch on" the viscous action. As a result of the internal friction forces the potential velocity field will be altered and becomes vortical. If in this process the fluid motion remains two-dimensional, the discontinuity radius $r_0$ will be unchanged. This is a result of the fact that in two-dimensional hydrodynamics of an incompressible fluid an infinite value of the energy is required for every expansion or contraction of a cavity [4].

In this formulation the problem is described by the system

$$\frac{\partial V_\varphi}{\partial t} = \nabla \left( \frac{\partial V_\varphi}{\partial r} + \frac{1}{r} \frac{\partial V_\varphi}{\partial \varphi} - \frac{V_\varphi}{r^2} \right),$$

$$\frac{\partial V_\varphi}{\partial r} - \frac{V_\varphi}{r} = 0 \quad \text{for} \quad r = r_0, \quad V_\varphi \to 0 \quad \text{for} \quad r \to \infty,$$

$$V_\varphi = \frac{r_0}{2\pi r} \quad \text{for} \quad t = 0. \quad (1)$$

Introducing the dimensionless quantities

$$U = \frac{V_\varphi}{V_\varphi^0}, \quad \tau = -\frac{\varphi}{V_\varphi^0}, \quad \xi = r/r_0, \quad V_\varphi = \frac{\Gamma}{2\pi r_0},$$

$$a = r_0 \frac{V_\varphi^0}{V} \left( \frac{r}{2\pi r_0} \right),$$

(here $V_\varphi^0$ is the initial value of the velocity at the boundary of the discontinuity) and using the new variable $\theta = U - a/\xi$, we obtain

$$\frac{\partial \theta}{\partial \tau} + \frac{1}{\xi} \frac{\partial \theta}{\partial \xi} - \frac{\theta}{\xi^2},$$

$$\frac{\partial \theta}{\partial \xi} = \frac{\theta}{a} \quad \text{for} \quad \xi = a, \quad \theta \to 0 \quad \text{for} \quad \xi \to \infty,$$

$$\theta = 0 \quad \text{for} \quad \tau = 0. \quad (2)$$

We shall use the Laplace transform method to solve the equation. After transformation, system (2) reduces to the Bessel equation for a function of imaginary argument $\Theta(p, \xi)$ is the transform of the function $\theta(\tau, \xi)$

$$\xi^2 \frac{\partial^2 \Theta}{\partial \xi^2} + \xi \frac{\partial \Theta}{\partial \xi} - \Theta(p^2 + 1) = 0, \quad (3)$$

with the boundary conditions

$$\partial \Theta/d\xi (a) = 2/a \quad \text{for} \quad \xi = a, \quad \Theta \to 0 \quad \text{for} \quad \xi \to \infty.$$

The general solution of the Bessel equation (3) has the form

$$\Theta(p, \xi) = C_1 I_1 (\sqrt{p^2 + 1}) + C_2 K_\xi (\sqrt{p^2 + 1}). \quad (4)$$

The MacDonald function $K_\xi$ satisfies the condition as $\xi \to \infty$, since $I_1 (\sqrt{p^2 + 1}) \to \infty \quad \text{as} \quad \xi \to \infty$; therefore

$$\Theta(p, \xi) = C K_\xi (\sqrt{p^2 + 1}) \quad (5)$$

Using the boundary condition for $\xi = a$, we obtain

$$C = \frac{2}{r_0 \sqrt{r_0^2 + |V_\varphi^0|^2}} \int_{-r_0}^{r_0} \frac{V_\varphi^0}{r_0} \, dp. \quad (6)$$

The MacDonald function $K_\xi (z)$ does not have roots in the right half-plane of the complex variable $z$ [5], where $|\arg z| \leq \pi/2$. To the set of points lying in the right half-plane of the complex variable $z$, for which

$$\frac{I_1 (\sqrt{p^2 + 1})}{\sqrt{p^2 + 1}}$$

This expression for $\Theta(p, \xi)$ is the Laplace transform of the function $\theta(\tau, \xi)$.

We find the original $V_\varphi(\tau, \xi)$, whose transform is given by the expression

$$\Theta(p, \xi) = p \Phi(p, \xi) = -\frac{2K_\xi (\sqrt{p^2 + 1})}{a \sqrt{p^2 K_\xi (\sqrt{p^2 + 1})}},$$

in accordance with the inversion equation formula for the Laplace transform

$$\Phi(\tau, \xi) = \frac{2}{a \sqrt{p^2 K_\xi (\sqrt{p^2 + 1})}}.$$
To find the solution of (5) we make use of the contour ABCDEF (Fig. 1). The integrand is single-valued everywhere on and within the contour and, according to the Cauchy theorem, the integral along ABCDEF is equal to zero.

Since in the limit with approach of the radius of the large circle to infinity, the integrals along BC and AF tend to zero, and the contribution from the integral along the small semicircle DE with center at the coordinate origin is also equal to zero for our case, then integral (5) is replaced by two nonsingular integrals along the negative real semiaxis, obtained from the integrals along CD and EF.

We take
\[ p = u^2 e^{-i\nu}, \quad \sqrt{\frac{p}{u^2}} = u e^{-i\nu}, \quad dp = -2udu \quad \text{along CD}, \]
\[ p = u^2 e^{i\nu}, \quad \sqrt{\frac{p}{u^2}} = u e^{i\nu}, \quad dp = -2udu \quad \text{along EF}; \]

then, using known formulas [6]
\[ K_\nu(z e^{\pm i\nu}) = -\frac{1}{2\pi} \int_0^\infty J_\nu(u z) - J_\nu(u z) J_\nu(u z) du, \]
\[ K_\nu(z e^{-i\nu}) = \frac{1}{2\pi} \int_0^\infty J_\nu(u z) + J_\nu(u z) J_\nu(u z) du, \]
we obtain from (5)
\[ q(\xi, \tau) = \frac{a}{2\pi} \int_0^\infty e^{-u^2} \frac{J_1(u \xi) Y_1(u \xi) - Y_1(u \xi) J_1(u \xi)}{J_3(u \xi) + Y_3(u \xi)} du. \]

According to the theorem on the transform of the integral of a function, we shall find the solution in the form
\[ \Theta(\tau, \xi) = \frac{a}{2\pi} \int_0^\infty e^{-u^2} \frac{J_1(u \xi) Y_1(u \xi) - Y_1(u \xi) J_1(u \xi)}{J_3(u \xi) + Y_3(u \xi)} du \times \frac{J_1(u \xi) Y_1(u \xi) - Y_1(u \xi) J_1(u \xi)}{J_3(u \xi) + Y_3(u \xi)} du, \]

The approximate solution for small times may be found by using the asymptotic expansion of the Bessel functions, and the formula for the transform \( \Theta(\tau, \xi) \) may be found in the form of a series in exponential functions [7], whose coefficients are terms of the series in \( 1/\sqrt{\tau} \).

Since
\[ K_\nu(z) = \left( \frac{\pi^{1/2}}{2z^{1/2}} \right) e^{-z} \left( 1 + \frac{4z^{1/4} - 1}{11.8z} + \frac{(4z^{1/4} - 1)(4z^{1/4} - 2)}{27(8z^4)} + O(x^3) \right), \]
then
\[ \Theta(\tau, \xi) = -\frac{a}{2} \left( \frac{\pi^{1/2}}{a} \right) e^{-\xi^2} \frac{1}{\sqrt{\tau}} \int_0^\infty \frac{d\tau}{\sqrt{\tau}} \left( 1 + \frac{2z - 15\xi}{8a \sqrt{\tau}} + \frac{15\xi + 90\xi^2 - 243 \xi^3}{128a^2 \sqrt{\tau}} \right) + \ldots. \]

Using the tabulated inversion formulas [7], we find for small \( \tau \)
\[ \Theta(\xi, \tau) = -\frac{2}{a} \left( \frac{\pi^{1/2}}{a} \right) \int_0^\infty \frac{\sqrt{\tau} J_0(\xi \sqrt{\tau})}{\sqrt{\tau} + \frac{2\chi}{3}} + \ldots. \]

In dimensional form the solution for the rate of rotation, suitable for small values \( vt/r_0^2 \), has the form
\[ V_\nu(r, t) = \frac{\Gamma}{2\pi r} \int_0^\infty \frac{\sqrt{\tau} J_1(\xi \sqrt{\tau})}{\sqrt{\tau} + \frac{2\chi}{3}} + \ldots \]

and for \( r = r_0 \) (on the discontinuity surface)
\[ V_\nu(r = r_0, t) = \frac{\Gamma}{2\pi r_0} \left( 1 - \frac{4}{3} \frac{\sqrt{\tau}}{r_0} - \frac{5(\nu \xi)^2}{2\sqrt{\tau}} + \ldots \right). \]

To find the asymptotic solution for large values of time we use the expansion of the Bessel functions in the form [6, 7]
\[ K_\nu(z) = (\sqrt{\pi} \frac{\nu^{1/2}}{2z^{1/2}}) e^{-z} \sum_{k=0}^{\infty} \frac{(\nu z)^{2k}}{k! (n+k)!} \times \sum_{m=1}^{\infty} \frac{(\nu z)^{2m} - 2^{2m} \nu^{2m} (n+k-1)!}{k! (n+k)!} \times \frac{\Gamma(n+k+1)}{k! (n+k+1)!}, \]
where \( \Gamma \) is the gamma function, and \( \nu = \ln C = 0.5772 \) is the Euler constant. Then
\[ \Theta(\xi, \tau) = -\frac{2}{a} \left( \frac{\pi^{1/2}}{a} \right) \int_0^\infty \frac{e^{\pi^2}}{\sqrt{\tau} + \frac{2\chi}{3}} \int_0^\infty \frac{d\tau}{\sqrt{\tau} + \frac{2\chi}{3}} \left( \frac{\pi^{1/2}}{a} \right) \left( 1 - \frac{4}{3} \frac{\sqrt{\tau}}{r_0} - \frac{5(\nu \xi)^2}{2\sqrt{\tau}} + \ldots \right). \]